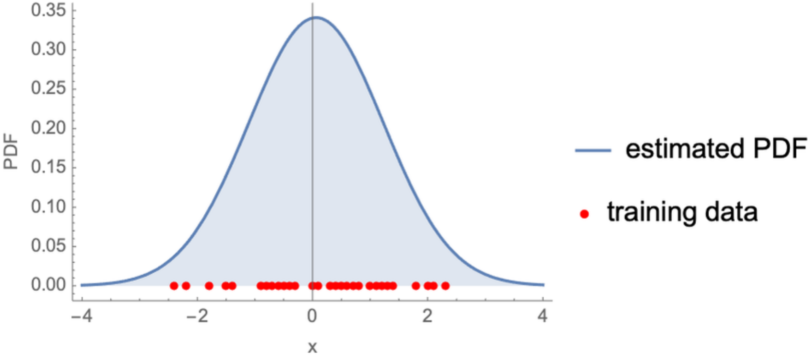
Logistic regression
DS351

Learning

Learning **probability distribution** from **data**



Many ways of learning

- Supervised learning $\leftarrow$ today's topic
- Unsupervised learning
- Semi-supervised learning
- Online learning
- Reinforcement learning
- and so on...

Supervised learning

Labeled data:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

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Goal: From these data, learn a function f that accurately maps x to y

$$f(x) = y$$

Supervised learning

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New data

$$x_{n+1}$$

What is the most likely label of y? Our prediction is $\hat{y} = f(x)$

Supervised learning tasks

So far, our tasks that we've covered can be framed as supervised learning tasks

- Regression: predict $y \in (-\infty, \infty)$ from $x_1, x_2, \ldots, x_p$

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- Regression: predict $y \in (-\infty, \infty)$ from $x_1, x_2, \ldots, x_p$
- Forecasting: predict y_{T+1} from $y_1, y_2, \ldots, y_T$

Supervised learning tasks

So far, our tasks that we've covered can be framed as supervised learning tasks

- Regression: predict $y \in (-\infty, \infty)$ from $x_1, x_2, \ldots, x_p$
- Forecasting: predict y_{T+1} from $y_1, y_2, \ldots, y_T$
- Classification: predict $y \in \{1, 2, \ldots\}$ from $x_1, x_2, \ldots, x_p$

Classification

Given features, want to predict **binary** or **categorical** variables

$$\text{features} \;\Rightarrow\; \text{label}$$

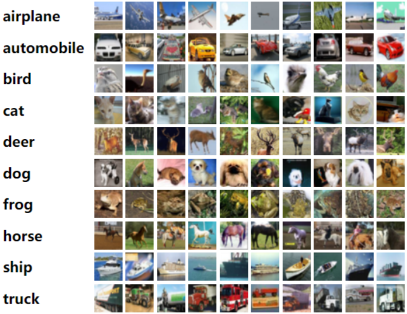
- $\{0, 1\}, \{\text{True}, \text{False}\}$
- $\{1, 2, \ldots, N\}$

Classification problems

Is this a **cat** or a **dog**?

(cat)

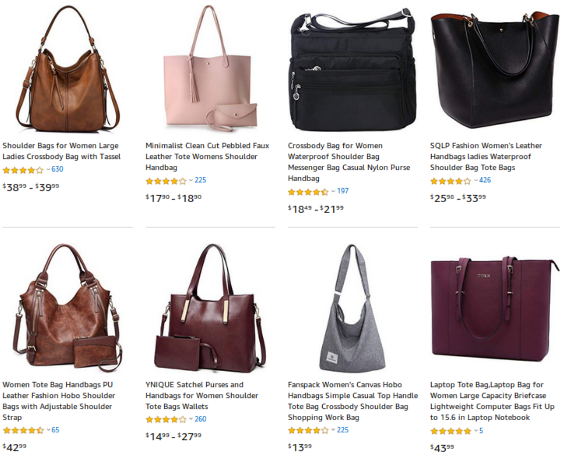
Classification problems



What is the object in a particular image?

(e.g. robot, automatic car)

Classification problems

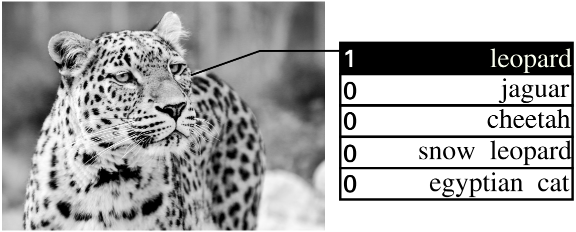


Will I **click on** these products?

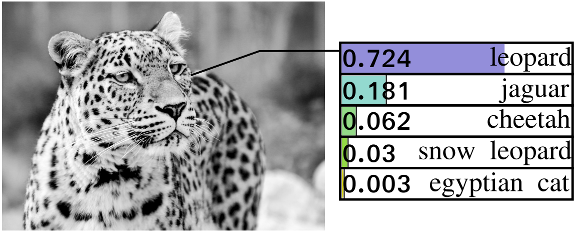
(no)

Probabilistic framework

Instead of directly predicting 0's and 1's

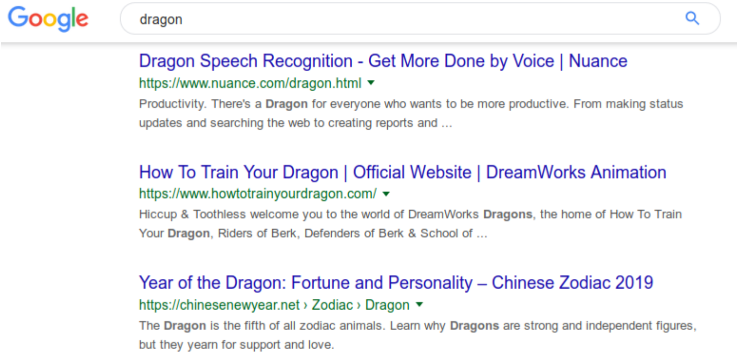


we could predict the probability of being in each class:



Applications

- **Ranking** of the search results by probabilities



- **Medical diagnosis**
  - Looking at the heart rate, blood pressure etc., what is the chance of contracting a heart disease?

Binary classification

Given: an instance with features $\boldsymbol{x}$ and possible label $y = 0$ or $y = 1$.
Goal: Predict the probability of the instance being in class 0 and 1:

$$P(y = 0|\boldsymbol{x}) \qquad \text{and} \qquad P(y = 1|\boldsymbol{x})$$

We then make the following prediction:

$$\boxed{\hat{y} = \begin{cases} \mathbf{0} & \text{if } P(y = 1|\boldsymbol{x}) \leq 0.5 \\ \mathbf{1} & \text{if } P(y = 1|\boldsymbol{x}) > 0.5 \end{cases}}$$

Multiclass classification

Given: an instance with features $\boldsymbol{x}$ and possible label $y = 1, 2, \ldots, N$.
Goal: Predict the probability of the instance being in class $1, 2, \ldots, N$:

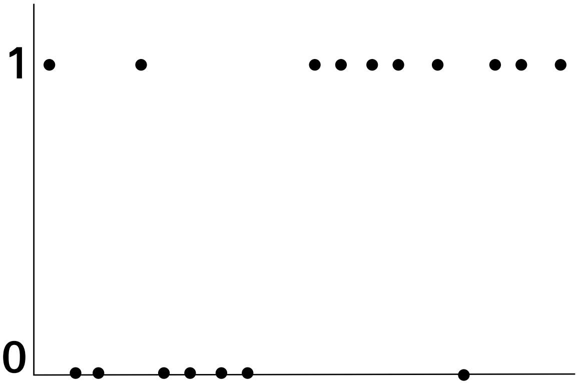
$$P(y = j|x) \quad \text{for } j = 1, 2, \ldots, N$$

We then make the following prediction:

$$\boxed{\hat{y} = J \text{ if } P(y = J|\boldsymbol{x}) > P(y = j|\boldsymbol{x}) \text{ for any other } j}$$

Predicting probability

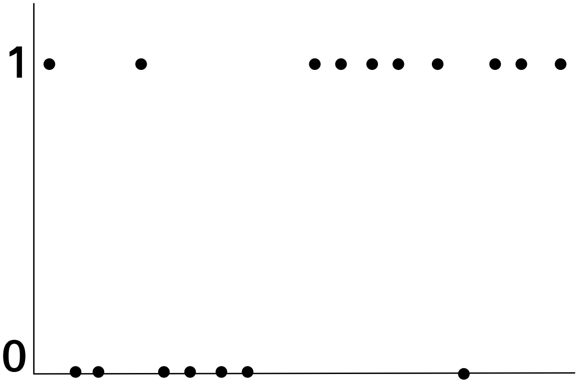
Can we use linear regression to do this?



We need some function that stays between 0 and 1.

Predicting probability

Instead, we need something like this:



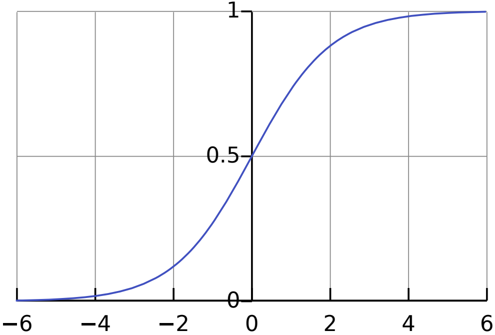
Logistic regression

That is, we are looking for a function with the following properties:

1. Stays between 0 and 1
2. Continuous
3. Symmetric

Logistic regression

Sigmoid function: $\sigma(x) = \frac{1}{1+e^{-x}}$



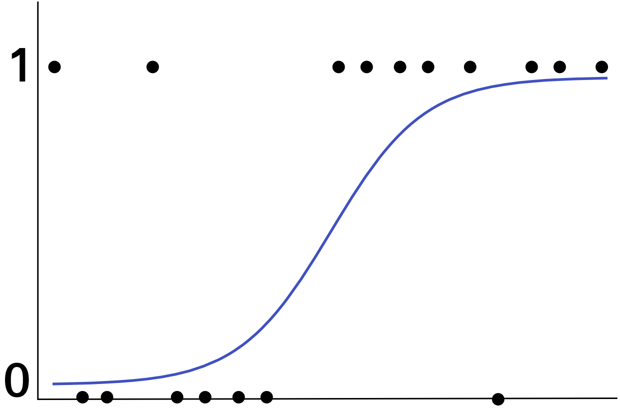
- If $x \to -\infty$ then $\sigma(x) \to 0$.
- If $x \to \infty$ then $\sigma(x) \to 1$.

Logistic regression

Find *coefficients* $A = [a_0, a_1, \ldots, a_m]$ such that

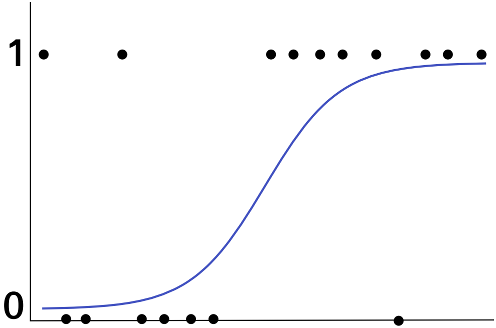
$$P(y = 1|x) = \frac{1}{1 + e^{-(a_0 + a_1 x_1 + \ldots + a_m x_m)}} = \frac{1}{1 + e^{-A \cdot \mathbf{x}}}$$

best fit the data



Logistic regression

$$P(y=1|x) = \frac{1}{1+e^{-(a_0+a_1x_1+\ldots+a_mx_m)}} = \frac{1}{1+e^{-A\cdot\mathbf{x}}}$$



- If $a_0 + a_1x_1 + \ldots + a_mx_m \to \infty$ then $\sigma(x) \to 1$.
- If $a_0 + a_1x_1 + \ldots + a_mx_m \to -\infty$ then $\sigma(x) \to 0$.

Logistic regression

Find *coefficients* $A = [a_0, a_1, \ldots, a_m]$ such that

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best fit the data

What is $P(y = 0|\boldsymbol{x})$?

Log-odds

How can we interpret the linear function $a_0 + a_1 x_1 + \ldots + a_m x_m$ in this model?

$$\log\left(\frac{P(y = 1 \mid \boldsymbol{x})}{P(y = 0 \mid \boldsymbol{x})}\right) =$$

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$$\log\left(\frac{P(y=1 \mid \boldsymbol{x})}{P(y=0 \mid \boldsymbol{x})}\right) =$$

- This is called **log-odds** or **logit**.
- Example: 1 unit increase in $x_1 \Rightarrow a_1$ unit increase in log-odds

Maximum-likelihood principle

Principle: If the data point (x, y) already appears in the data, then the probability $P(y|x)$ is high.

= $[x_1, x_2, \ldots, x_{784}]$

Goal: **Maximize** the probability $P(y|x)$ **for all** data points (x, y).

Maximum-likelihood principle

Given data:

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Likelihood = Probability that the data is generated from our model

$$L(A) = P(y^{(1)}|x^{(1)})P(y^{(2)}|x^{(3)}) \ldots P(y^{(n)}|x^{(n)})$$

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Find $A = [a_0, a_1, a_2, \ldots, a_m]$ that maximizes $L(A)$

Example: Credit card data

Is the user going to default on their credit card?
$y = 1$: default, $y = 0$: not default

	Coefficient	Std. error	Z-statistic	P-value
Intercept	-10.8690	0.4923	-22.08	< 0.0001
balance	0.0057	0.0002	24.74	< 0.0001
income	0.0030	0.0082	0.37	0.7115
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- 1 baht increase in balance = 0.0057 unit increase in log-odds
- $Z = \frac{\hat{\beta}_i}{\text{SE}(\hat{\beta}_i)}$.
- $H_0 : \beta_1 = 0$ is rejected; there is an association between `balance` and the probability of default

Predictions

Comparing card defaulting of student and non-student

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$$\hat{p}(y = 1|x_1 = 1,500, x_2 = 40, x_3 = 1) = \frac{1}{1 + e^{-(-10.869+0.00574\times 1{,}500+0.003\times 40-0.6468\times 1)}} = 0.058$$

$$\hat{p}(y = 1|x_1 = 1,500, x_2 = 40, x_3 = 0) = \frac{1}{1 + e^{-(-10.869+0.00574\times 1{,}500+0.003\times 40-0.6468\times 0)}} = 0.105.$$

Non-students have higher chance of defaulting their cards.

Framingham dataset

- Label: Diagnosed with a heart disease in the next 10 years
- Features: gender, smoking, blood pressure, heart rate, blood sugar, cholesterol, BMI

The model

$$P(y = 1|\text{CigsPerDay, Chol, BMI ...}) \\ = \frac{1}{1 + e^{-(0.04\text{CigsPerDay}+0.002\text{Chol}+0.003\text{BMI}+...)}}.$$

- If $P(y = 1|\text{CigsPerDay, Chol, BMI ...}) = 0.2 \Rightarrow$, classify y as 0
- If $P(y = 1|\text{CigsPerDay, Chol, BMI ...}) = 0.8 \Rightarrow$ classify y as 1
- With everything else fixed, higher CigsPerDay $\Rightarrow$ higher chance of heart disease.
- $+1$ cigarette per day $= +0.04$ log-odds.

Cross-validation accuracy

$$\text{Accuracy} = \frac{\#\text{Correctly classified}}{\#\text{Total}}$$

Evaluation by train-test split

- Split data a **training set** and **test set**
- **Train** the model on the training set
- Computing the accuracy of the model's predictions on the test set

	1NN	3NN	5NN	7NN	9NN	Logistic
Accuracy	77.55	81.96	83.18	83.96	84.29	85.40

Multiclass logistic regression

N-class classification

Data:

$$(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(n)}, y^{(n)}), \quad y \in \{1, 2, \ldots, N\}$$

Multiclass logistic regression

N-class classification

Data:

$$(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(n)}, y^{(n)}), \quad y \in \{1, 2, \ldots, N\}$$

Model parameters: $N-1$ vectors $A_1, A_2, \ldots, A_{N-1}$

$$P(y=1|\boldsymbol{x}) = \frac{e^{A_1 \cdot \boldsymbol{x}}}{1 + \sum_{i=1}^{n-1} e^{A_i \cdot \boldsymbol{x}}}$$

$$P(y=2|\boldsymbol{x}) = \frac{e^{A_2 \cdot \boldsymbol{x}}}{1 + \sum_{i=1}^{n-1} e^{A_i \cdot \boldsymbol{x}}}$$

$$\cdots$$

$$P(y=N-1|\boldsymbol{x}) = \frac{e^{A_{N-1} \cdot \boldsymbol{x}}}{1 + \sum_{i=1}^{n-1} e^{A_i \cdot \boldsymbol{x}}}$$

$$P(y=N|\boldsymbol{x}) = \frac{1}{1 + \sum_{i=1}^{n-1} e^{A_i \cdot \boldsymbol{x}}}$$

Example

When we use the model after training:

$$\boldsymbol{x} = (25, 10, 0.5, 82)$$

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$$\boldsymbol{x} = (25, 10, 0.5, 82)$$

- If

  $$P(y = 1|\boldsymbol{x}) = 0.3, P(y = 2|\boldsymbol{x}) = 0.3, P(y = 3|\boldsymbol{x}) = 0.4$$

  classify $y = 3$.

Example

When we use the model after training:

$$\boldsymbol{x} = (25, 10, 0.5, 82)$$

- If

  $$P(y = 1|\boldsymbol{x}) = 0.3, P(y = 2|\boldsymbol{x}) = 0.3, P(y = 3|\boldsymbol{x}) = 0.4$$

  classify $y = 3$.
- If

  $$P(y = 1|\boldsymbol{x}) = 0.2, P(y = 2|\boldsymbol{x}) = 0.4, P(y = 3|\boldsymbol{x}) = 0.4$$

  randomly pick $y = 2$ or $y = 3$.