Mathematical Background 1

Source: Ahmadi, ORF 363 slides.

Inner products

Vector norms

Matrix norms

Continuity

Linear, Affine and Quadratic functions

Inner product

An **Inner Product** is a real-valued function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that satisfies the following properties:

- Positivity: $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$
- Symmetry: $\langle x, y \rangle = \langle y, x \rangle$
- Additivity: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- Homogeneity: $\langle rx, y \rangle = r \langle x, y \rangle$ for all $r \in \mathbb{R}$

Example 1: Classical multiplication

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ \langle x, y \rangle = x \cdot y \end{aligned}$$

Check that this is an inner product.

Example 2

$$
\begin{aligned}
\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R} \\
\langle x, y \rangle &= x_1 y_1 + x_2 y_2 \\
&= \text{length}(x) \cdot \text{length}(y) \cdot \cos(\theta)
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The inner product is

- positive when $\theta < 90$
- negative when $\theta > 90$
- zero when $\theta = 90$

Euclidean inner product

The two previous examples are $n = 1$ and $n = 2$ cases of the **Euclidean inner product**:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i = x^T y \quad \text{where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

Check that this is an inner product

Orthogonality

Two vectors x and y are **orthogonal** if $\langle x, y \rangle = 0$

- With this definition the zero vector is orthogonal to every other vector.
- But two nonzero vectors can also be orthogonal.
    - For example,

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, y = \begin{pmatrix} 3 \\ -\frac{3}{2} \end{pmatrix}$$

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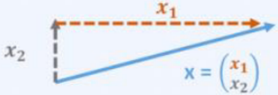
Norms

A vector norm is a real valued function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following properties:

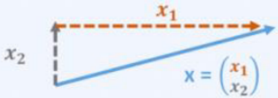
- Positivity: $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$
- Homogeneity: $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{R}$
- Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$

Examples of norms

Figure			
Name	1-norm or $\|\cdot\|_1$	2-norm or $\|\cdot\|_2$ or Euclidean norm	∞-norm or $\|\cdot\|_\infty$
Definition	$\|x\|_1 = \|x_1\| + \cdots + \|x_n\|$	$\|x\|_2 = \sqrt{< x, x >} = \sqrt{x_1^2 + \cdots + x_n^2}$	$\|x\|_\infty = \max_i \|x_i\|$
On the figure	$\|x\|_1 = \|x_1\| + \|x_2\| = \rightarrow + \uparrow$	$\|x\|_2 = \sqrt{x_1^2 + x_2^2} = \nearrow$	$\|x\|_\infty = x_1 = \rightarrow$

- Check that these are norms

Examples of norms

Figure			
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- Notation: $\|\cdot\|$ is the Euclidean norm ($\|\cdot\|_2$)

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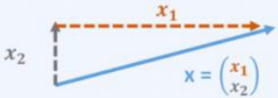
Figure			
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- Check that these are norms
- Notation: $\|\cdot\|$ is the Euclidean norm ($\|\cdot\|_2$)
- $\|x\| \geq \|x\| \geq \|x\|$

Norm from inner product

Given any inner product $\langle x, y \rangle$, one can construct a norm given by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Cauchy Schwarz Inequality

For any two vectors x and y in $\mathbb{R}^n$, we have the **Cauchy-Schwarz inequality**:

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

Furthermore, equality holds iff $x = ry$ for some $r \in \mathbb{R}$.

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Matrix norm

Matrix norms $\|\cdot\| : \mathbb{R}^{m \times n} \to \mathbb{R}$ are functions that satisfy exactly the same properties as in the definition of a vector norm.

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- Homogeneity: $\|rA\| = |r|\|A\|$ for all $r \in \mathbb{R}$
- Triangle inequality: $\|A + B\| \leq \|A\| + \|B\|$

Example: Induced norm

For any vector norm $\|\cdot\|_* : \mathbb{R}^k \to \mathbb{R}$, the **Induced Norm** $\|\cdot\|_* : \mathbb{R}^{m\times n} \to \mathbb{R}$ on the space of $m \times n$ matrices is defined as:

$$\|A\|_* = \max\{\|Ax\| : x \in \mathbb{R}^n \text{ and } \|x\|_* = 1\}$$

Example: Frobenius norm

The **Frobenius norm** $\|\cdot\|_F : \mathbb{R}^{m\times n} \to \mathbb{R}$ is defined by:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$$

Example

The $n \times n$ identity matrix

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 1 \end{pmatrix}$$

satisfies $I_n x = x$ for any vector $x \in \mathbb{R}^n$

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For any vector norm $\|\cdot\|_*$, the induced norm of I_n

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Example

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satisfies $I_n x = x$ for any vector $x \in \mathbb{R}^n$

$$\|I_n\|_F = \sqrt{1^2 + 0^2 + \ldots + 1^2} = \sqrt{n}$$

Submultiplicative norm

A matrix norm is **submultiplicative** if it satisfies the following inequality:

$$\|AB\| \leq \|A\| \cdot \|B\|$$

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- All induced norms are submultiplicative
- .The Frobenius norm is submultiplicative
- The norm $\|\cdot\| : A \to \max_{i,j}|a_{ij}|$ is **not** submultiplicative

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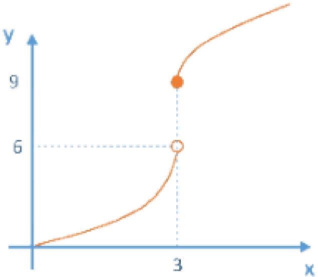
Continuity

A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at a point** $a \in \mathbb{R}$ if, as $x \to a$, $f(x) \to f(a)$.



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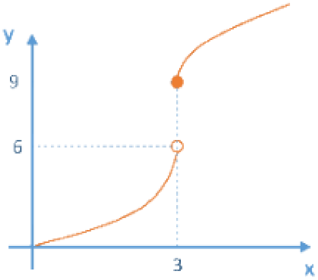

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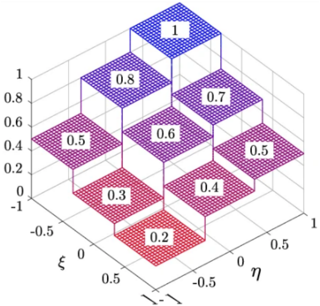
A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **continuous at a point** $a \in \mathbb{R}^n$ if, as $x \to a$, $f(x) \to f(a)$.



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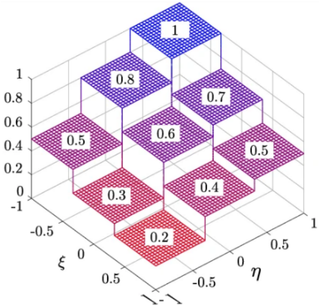

Continuity

- Hard to check continuity in higher dimensions since we have to check all paths to $a \in \mathbb{R}^n$.

- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ given as $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$ is continuous iff each entry $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuous.

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- **Example:** $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x_1, x_2, x_3) = \begin{pmatrix} x_1 x_2 \\ \cos(x_3) \\ \log(x_1^2 + 1) \end{pmatrix}$ is continuous.

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Linear function

A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if:

- $L(\alpha x) = \alpha L(x)$ for all $x \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$
- $L(x + y) = L(x) + L(y)$ for all $x, y \in \mathbb{R}$

Linear function

Any linear function can be represented as

$$L(x) = Ax \qquad \text{for some } m \times n \text{ matrix } A$$

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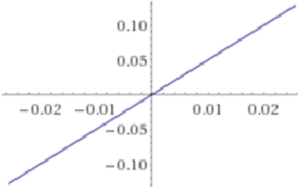
For the special case $m = 1$, linear functions take the form

$L(x) = a^T x$ for some vector $a \in \mathbb{R}^n$.

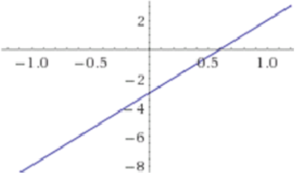
Affine function

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is **affine** if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}^m$ and a vector $y \in \mathbb{R}^m$ such that:

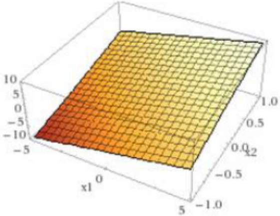
$$f(x) = L(x) + y \qquad \text{for all } x \in \mathbb{R}^n$$



Linear $m = n = 1$



Affine $m = n = 1$



Linear $n = 2, m = 1$

Quadratic function

A **quadratic form** $f : \mathbb{R}^n \to \mathbb{R}$ is a function that can be represented as

$$f(x) = x^T Q x$$

where Q is an $n \times n$ matrix that we can assume to be symmetric.

Example

- $n = 1$, $Q = 2$.

- $n = 2$, $Q = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

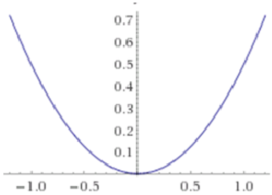
Why can we assume that Q is symmetric?

Example: Let's look at $Q = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix}$

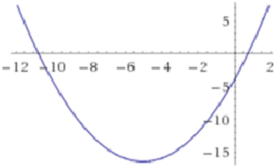
Quadratic function

A quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ is a function that is the sum of a quadratic form and an affine function:

$$f(x) = x^T Q x + a^T x + b$$



Quadratic form $n = 1$



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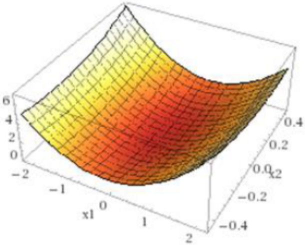

Quadratic form $n = 2$