Optimization problems – basic notations

Unconstrained optimization

First order condition for optimality

Second order conditions for optimality

Notation reminder

$$f(x) = f(x_1, x_2, \ldots, x_n) \qquad (f : \mathbb{R}^n \to \mathbb{R})$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} \quad \nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n^2}(x) \end{pmatrix}$$

Gradient vector

Hessian matrix

First order condition

Theorem. **(First Order Necessary Condition for (Local) Optimality)**
If x^* is an unconstrained local minimizer of a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, then we must have:

$$\nabla f(x^*) = 0$$


Fermat
(1607-1665)

Remarks

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- Nevertheless, it is useful because any local minimum must satisfy this condition. So, we can look for local (or global) minima only among points that make the gradient of the objective function vanish.
- Terminology: A point x that satisfies $\nabla f(x) = 0$ is called a **stationary point** or a **critical point** of f. stationary point or a critical point of

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Linear algebra review

Symmetric matrix: $A = A^T$

Theorem. Eigenvalues of a real symmetric matrix are real.

A square matrix A is said to be

- Positive semidefinite (psd) if:

- Positive definite (pd) if:

Linear algebra review

Recall that when we talk of psd or pd, we can assume that our matrix is symmetric: If A was not symmetric, we can replace A by $\frac{A^T + A}{2}$ as:

$$x^T A x = x^T \left(\frac{A^T + A}{2} \right) x$$

Linear algebra review

Theorem. A matrix is positive semidefinite iff all its eigenvalues are nonnegative. A matrix is positive definite iff all its eigenvalues are positive.

Examples:

1. $A = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$
2. $A = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$
3. $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

Second order condition

Theorem. (Second Order Necessary Condition for (Local) Optimality)
If x^* is an unconstrained local minimizer of a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, then, in addition to $\nabla f(x^*) = 0$, we must have:

$$\nabla^2 f(x^*) \succeq 0$$

(i.e., the Hessian at x^* is positive semidefinite.)

Second order condition

Theorem. (Second Order sufficient Condition for (Local) Optimality)
Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, $\nabla f(x^*) = 0$ and

$$\nabla^2 f(x^*) \succ 0$$

(i.e. the Hessian at x^* is positive definite), then x^* is a strict local minimum of f.

Remarks

- $\nabla f(x^*) = 0$, $\nabla^2 f(x^*) \succeq 0$ is not sufficient for local optimality.

$$f(x) = x^3$$

- $\nabla^2 f(x) \succ 0$ is not necessary for (even strict global) optimality

$$f(x) = x^4$$

What are the questions in practice?

- How would we use all these optimality conditions to find local solutions and certify their optimality?

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- How would we use all these optimality conditions to find local solutions and certify their optimality?
- Is it easy to find points satisfying these conditions? e.g., is it easy to solve $\nabla f(x) = 0$?
- Suppose you found that a given point is locally optimal, how would you go about checking if it is also globally optimal?

Exercise: State the analogues of the three theorems for local maxima.

Example 1: Least squares

Given: A $m \times n$ matrix (columns of A are independent)

b $m \times 1$ vector

Solve: $\min_{x \in \mathbb{R}^n} \|Ax - b\|^2$

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Steps:

1. Write $f(x) = \|Ax - b\|^2$ as inner product of vectors
2. Solve $\nabla f(x) = 0$ for x
3. Show that $\nabla^2 f(x) = 2A^T A \succ 0$

Example 2

Find all the local minima and maxima of the following function:

$$f(x) = \frac{1}{2}x_1^2 + x_1x_2 + 2x_2^2 - 4x_1 - 4x_2 - x_2^3$$

$\nabla f(x) =$

$\nabla f(x) = 0 \Rightarrow$

$$f(x) = \frac{1}{2}x_1^2 + x_1x_2 + 2x_2^2 - 4x_1 - 4x_2 - x_2^3$$

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Convex optimization I

Source: Ahmadi, ORF 363 slides.

Convex optimization

Convex sets

Intersection and Union of convex sets

General form of optimization

$$\min f(x)$$

$$\text{subject to } x \in \Omega$$

- We consider a very important special case of constrained optimization problems known as **convex optimization problems**

- For these problems,
  - f will be a **convex function**
  - Ω will be a **convex set**

Why convex optimization?

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- Numerous important optimization problems in engineering, operations research, machine learning, etc. are convex

Convex optimization

Convex sets

Intersection and Union of convex sets

Convex set

A set $\Omega \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in \Omega$, the line segment connecting x and y is also in Ω. In other words,

$$x, y \in \Omega, \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda) y \in \Omega$$

A point of the form $\lambda x + (1 - \lambda) y, \lambda \in [0, 1]$ is called a **convex combination** of x and y

Examples

Convex:

Not convex:

Common convex sets in optimization

- **Hyperplanes**: $\{x \mid a^T x = b\}$ $(a \in \mathbb{R}^n, b \in \mathbb{R})$

- **Halfspaces**: $\{x \mid a^T x \leq b\}$ $(a \in \mathbb{R}^n, b \in \mathbb{R})$

Common convex sets in optimization

- **Euclidean balls**: $\{x \mid \|x - x_c\| \leq r, x_c \text{ fixed}\}$ $(x, x_c \in \mathbb{R}^n, r \in \mathbb{R})$

- **Ellipsoids**: $\{x \mid (x - x_c)^T P (x - x_c) \leq r\}$ $(x, x_c \in \mathbb{R}^n, r \in \mathbb{R}, P \succ 0)$

Fancier convex sets

- The set of positive semidefinite matrices:

$$S_+^{n\times n} = \{P \in S^{n\times n} \mid P \succ 0\}$$

Proof:

Fancier convex sets

- The set of nonnegative polynomials in n variables.

$$\mathbb{P}_n = \{p(x) \text{ polynomial} \mid p(x) \geq 0, \forall x \in \mathbb{R}^n\}$$

Proof:

Convex optimization

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Intersection and Union of convex sets

Intersections of convex sets

Intersection of two convex sets is convex:

$$\Omega_1 \text{ convex}, \Omega_2 \text{ convext} \Rightarrow \Omega_1 \cap \Omega_2 \text{ convex}$$

Proof:

Polyhedra

A polyhedron is the solution set of finitely many linear inequalities—ubiquitous in optimization theory

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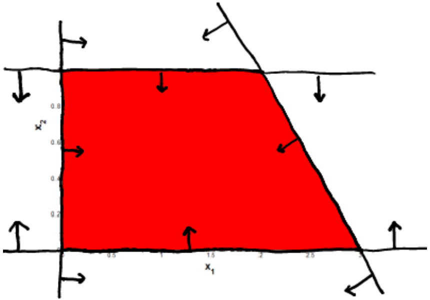
Example: Buying two products: prices are 50 and 200 bahts

$x_1 =$ quantity of the first item bought
$x_2 =$ quantity of the second item bought



Polyhedra

- In matrix form:

$$\{x \mid Ax \leq b\}$$

  where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$

- These sets are convex: intersection of halfspaces $a_i^T x \leq b_i$ where a_i is the i-th row of A

Union of convex sets

Obviously, the union may not be convex: