Convex optimization II

Source: Ahmadi, ORF 363 slides.

Convex functions

Epigraph and sublevel sets

Convex optimization problems

Convex function

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if its domain is a convex set and for all x, y in its domain, and all $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Convex function

Taking $\lambda = \frac{1}{2}$,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

Other definitions

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is

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- **Strictly convex** if $\forall x, y, x \neq y \forall \lambda \in (0, 1)$

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Note f is (strictly) concave iff $-f$ is (strictly) convex

convex
(and strictly convex)

concave
(and strictly concave)

neither convex nor concave

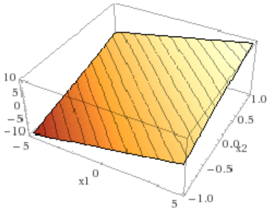
both convex and concave (but not strictly)

Examples of univariate convex functions

- e^{ax}
- $-\log x$
- x^a, $x \geq 0$, $a \geq 1$ or $a \leq 0$
- $-x^a$, $0 \leq a \leq 1$
- $|x|$, $a \geq 1$
- $x \log x$

Examples of convex functions

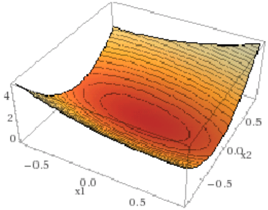
Affine function $f(x) = a^T x + b$ ($a \in \mathbb{R}^n, b \in \mathbb{R}$)



Examples of convex functions

Some quadratic functions

$$f(x) = x^T Q x + c^T x + d$$

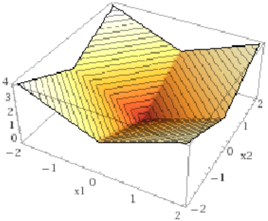


- Convex iff $Q \succeq 0$
- Strictly convex iff $Q \succ 0$
- Concave iff $Q \preceq 0$, Strictly concave iff $Q \prec 0$

Examples of convex functions

Any norm: meaning, any function $\|\cdot\|$ satisfying:

- $\|ax\| = |a|\|x\|$
- $\|x + y\| \leq \|x\| + \|y\|$
- $\|x\| \geq 0$ and if $\|x\| = 0 \Rightarrow x = 0$



Midpoint convex function

Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ is **midpoint convex** if its domain is a convex set and for all x, y in its domain, we have

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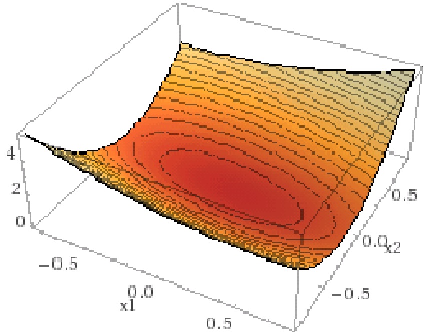
- Obviously, all convex functions are midpoint convex
- Continuous, midpoint convex functions are convex

Convexity $=$ Convexity along all lines

Theorem. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex iff the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(t) = f(x + ty)$$



Convexity $=$ Convexity along all lines

- Many of the algorithms we will see in future work by iteratively minimizing a function over lines

- It’s useful to remember that the restriction of a convex function to a line remains convex.

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Is there a connection between convex sets and convex functions?

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$$\text{epi}(f) = \{(x, t) \mid x \in \text{ domain}(f), f(x) \leq t\}$$

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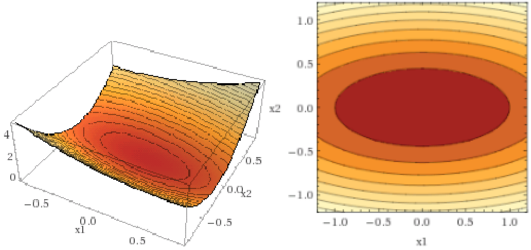
Theorem. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex iff its epigraph is convex

Convexity of sublevel sets

Definition. The α -sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$S_\alpha = \{x \in \text{domain}(f) \mid f(x) \leq \alpha\}$$



Convexity of sublevel sets

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Theorem. If a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then all its sublevel sets are convex sets

- Converse not true
- A function whose sublevel sets are all convex is called **quasiconvex**

Proof:

Convex functions

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Convex optimization problems

Convex optimization problems

A convex optimization problem is an optimization problem of the form

$$\begin{aligned} \min\ & f(x) \\ \text{s.t.}\ & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_j(x) = 0, \quad j = 1, \dots, k \end{aligned}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $h_i : \mathbb{R}^n \to \mathbb{R}$ are affine functions

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The **feasible set** $\Omega = \{x \in \mathbb{R}^n \mid g_i(x) \le 0, h_j(x) = 0\}$ is a convex set

Convex optimization problems

$$\min f(x)$$

$$\text{s.t. } g_i(x) \leq 0, \quad i = 1, \ldots, m$$

$$h_j(x) = 0, \quad j = 1, \ldots, k$$

Acceptable constraints

- Convex $\leq$ Concave
- Affine $=$ Affine
- Affine $\leq$ 0
- Convex $\leq$ Affine
- but not Strictly Convex = 0