Geometry and Simplex method

Source: Ahmadi, ORF 363 slides.

Solving an LP geometrically

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\begin{aligned}
\text{max} \quad & x_1 + 6x_2 \\
\text{s.t} \quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{aligned}
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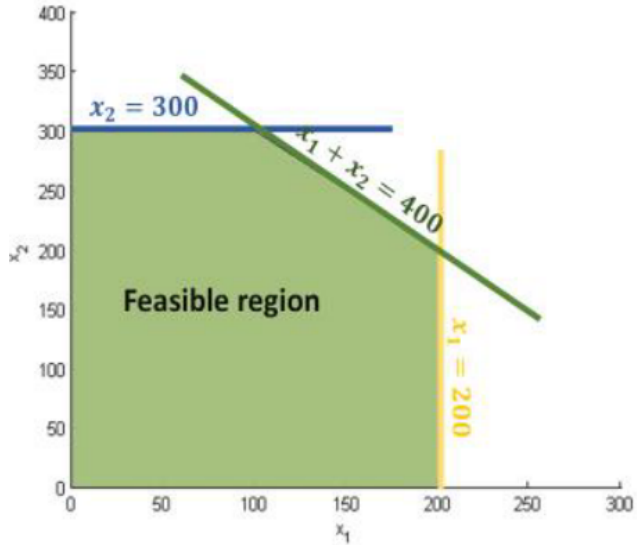

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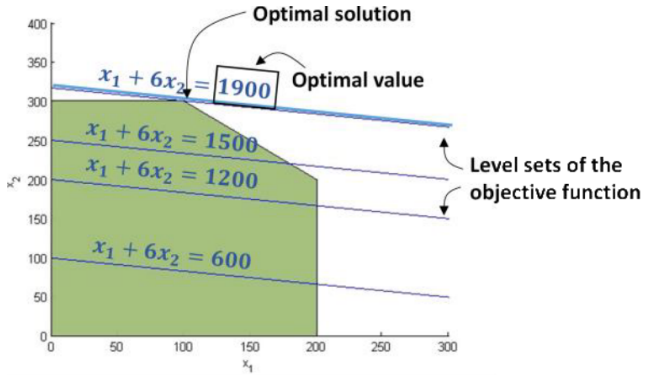


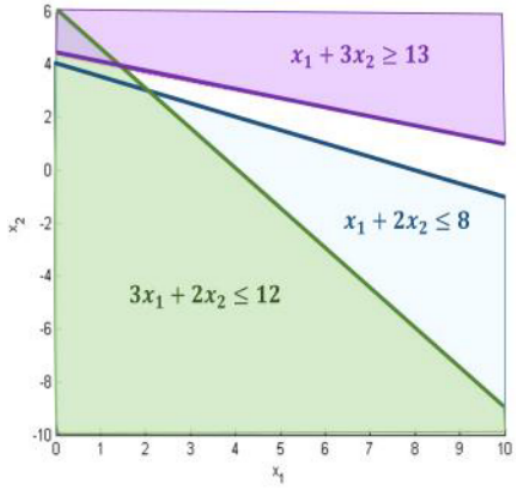
We are trying to find the ”largest” level set of the objective function that still intersects the feasible region

All possibilities for an LP

Infeasible

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t} \quad & x_1 + 2x_2 \leq 8 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1 + 3x_2 \geq 13 \end{aligned}$$

The three regions in the drawing do not intersect : there are no feasible solutions

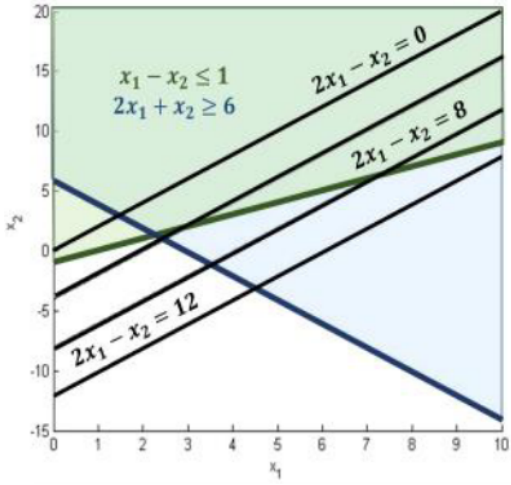


All possibilities for an LP

Unbounded

$$\begin{aligned} \min \quad & 2x_1 - x_2 \\ \text{s.t} \quad & x_1 - x_2 \leq 1 \\ & 2x_1 + x_2 \geq 6 \end{aligned}$$

The intersection of the green and the blue regions is unbounded; we can push the objective function as high up as we want



All possibilities for an LP

Infinite number of solutions

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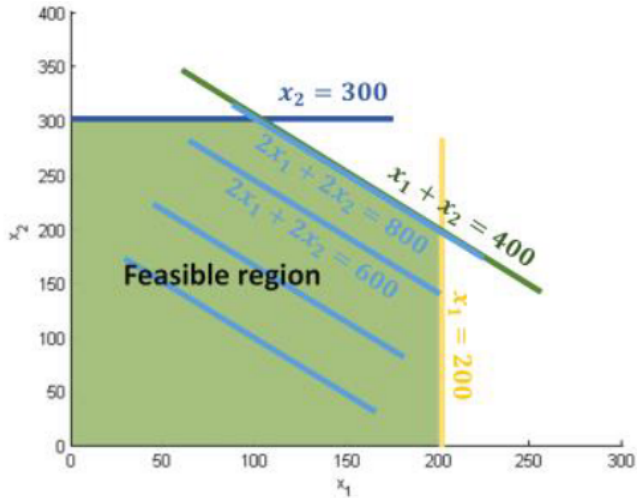
Entire “face” of the feasible region that is optimal.



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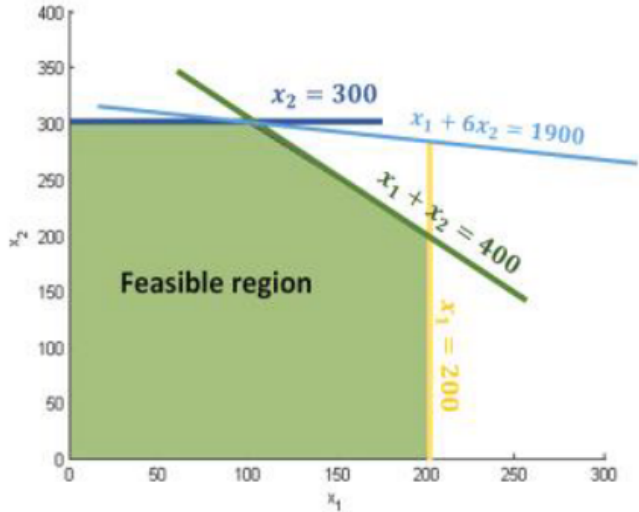

Geometry of LP

We will talk about the simplex algorithm, which exploits the geometry of LP. We'll go over some basic geometric results that are essential to this algorithm

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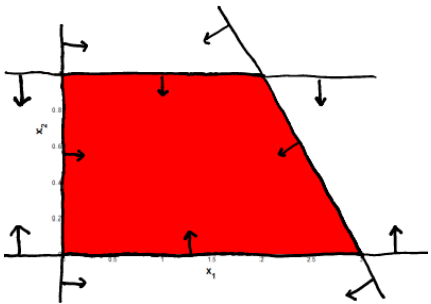
- **Hyperplanes**: $\{x \mid a^T x = b\}$ $(a \in \mathbb{R}^n, b \in \mathbb{R})$

- **Halfspaces**: $\{x \mid a^T x \leq b\}$ $(a \in \mathbb{R}^n, b \in \mathbb{R})$

Polyhedron

The intersection of finitely many half spaces is called a **polyhedron**

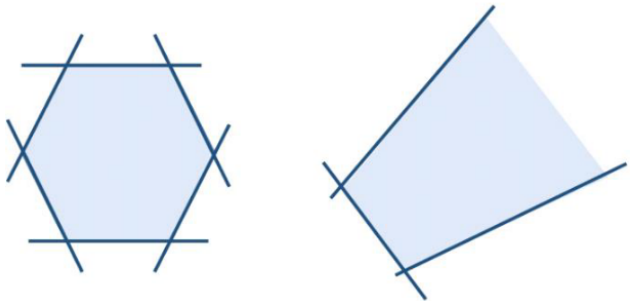


This is always a convex set

Polytope

- A set $S \subset \mathbb{R}^n$ is bounded if $\|x\| \leq K$ for some K for all $x \in S$

- A bounded polyhedron is called a **polytope**



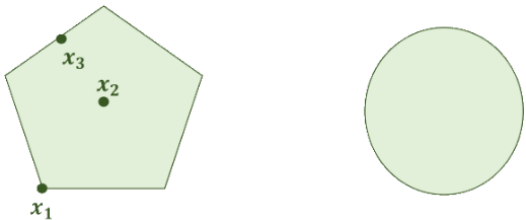
Extreme points

Let P be a convex set

- A point x is an **extreme point** of P if it cannot be written as a convex combination of two other points in P:

$$\text{No } y \neq x, z \neq x \text{ such that } x = \lambda y + (1-\lambda)z$$

- In other words, x does not lie on a segment joining two points in P



which point is extreme?

Vertex

Suppose we have one of the following constraints:

$$a^T x \geq b$$
$$a^T x \leq b$$
$$a^T x = b$$

A constraint is **tight** (or active or binding) at $\bar{x}$ if $a^T \bar{x} = b$

Equality constraints are tight by definition.

Vertex

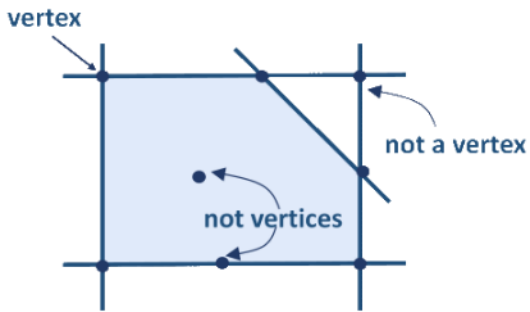
Constraints $a_1^T x \_\_ b_1, a_2^T x \_\_ b_2, \ldots, a_n^T x \_\_ b_n$ are **linearly independent** if the vectors $a_1, a_2, \ldots, a_n$ are independent

With these two definitions, we can now define the notion of a vertex of a polyhedron

Vertex

A point $x \in \mathbb{R}^n$ is a vertex of a polyhedron P, if

- $x \in P$
- There exists n independent constraints that are tight at x



Extreme point $\iff$ Vertex

Theorem. Let P be a polyhedron and $x \in P$, then

$$x \text{ is an extreme point} \iff x \text{ is a vertex}$$

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- The **vertex** definition is more useful for algorithmic purposes and is crucial to the simplex algorithm
- the **extreme point** definition gives the geometric intuition, and is used to prove the correctness of the simplex algorithm

Extreme point $\iff$ Vertex

Corollary. Given m constraints on points in $\mathbb{R}^n$, there can only be a finite number of extreme points

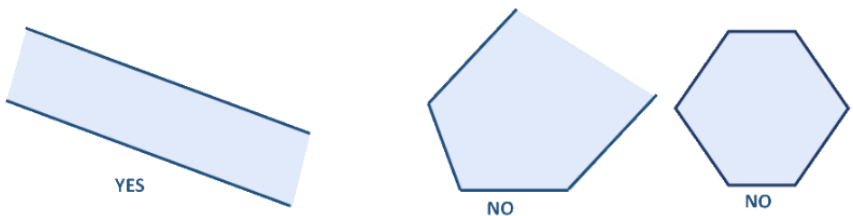
Proof.

$$
\begin{aligned}
a_1^T x &\geq b_1 \\
a_2^T x &\geq b_2 \\
&\vdots \\
a_m^T x &\geq b_m
\end{aligned}
$$

Existence of extreme points

A polyhedron contains a line if there is a point $x \in P$ and a direction $d \in \mathbb{R}^n$ such that

$$x + \lambda d \text{ is inside } P \quad \text{for all } \lambda \in \mathbb{R}$$



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2. P has at least one extreme point

Corollary. Every bounded polyhedron (i.e., every polytope) has an extreme point

Optimality of extreme points

Theorem. If the following LP

$$\begin{aligned} \min \ & c^T x \\ \text{s.t. } & Ax \geq b \quad (P) \end{aligned}$$

has at least one extreme point and at least one optimal solution, then there is at least one optimal solution at a vertex of P

Conclusion

When looking for an optimal solutionof an LP, it is enough to examine only the extreme points

The simplex algorithm

Main idea Consider an LP:

$$\begin{aligned} \max \ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

In a nutshell, this is all the simplex algorithm does:

- start at a vertex
- while there is a better neighboring vertex, move to it

Neighboring vertex

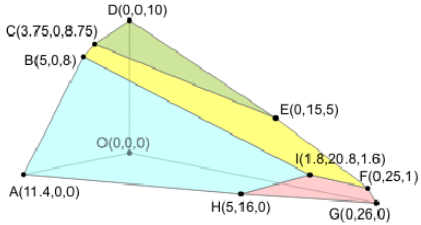
Two vertices are **neighbors** if they share tight constraints

Example:

$$
\begin{aligned}
\text{max} \quad & 20x_1 + 10x_2 + 15x_3 & \\
\text{s.t.} \quad & 3x_1 + 2x_2 + 5x_3 \leq 55 & (1)\\
& 2x_1 + x_2 + x_3 \leq 26 & (2)\\
& x_1 + x_2 + 3x_3 \leq 30 & (3)\\
& 5x_1 + 2x_2 + 4x_3 \leq 57 & (4)\\
& x_1, x_2, x_3 \geq 0 & (5,6,7)
\end{aligned}
$$

Neighboring vertex

Here's the feasible set:



Points $A, \ldots, I$ are vertices of the polyhedron. Consider $B = (5, 0, 8)$. The tight constraint of B are: constraints here are

- (6) as $x_2 = 0$
- (1) as $3 \cdot 5 + 2 \cdot 0 + 5 \cdot 8 = 55$
- (4) as $5 \cdot 5 + 2 \cdot 0 + 4 \cdot 8 = 57$

Neighboring vertex

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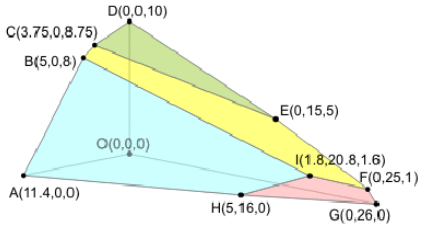

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Points A and B are neighbors as they both have constraints (6) and (4) tight, but A has (5) tight while B has (1) tight

The simplex algorithm

At every iteration of the simplex algorithm, we complete two tasks:

1. Check whether the current vertex is optimal (if yes, we're done)
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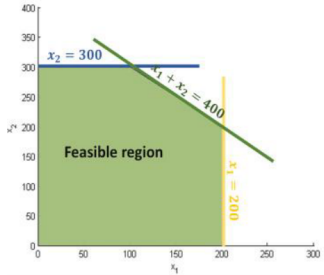
We start the algorithm at the origin

- The origin is optimal if and only if $c_i < 0$ for all i
  - If some $c_i > 0$, we can increase x_i until a new constraint becomes tight



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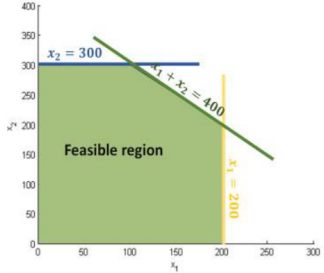

- Once at a new vertex, we move it to the origin by a "change of coordinates". Then we simply repeat.