Duality

Source: Ahmadi, ORF 363 slides.

Simplex method: an example

Duality

Simplex method: example

Let's try to solve the following LP: $200 - x_1 - x_2$ $100 - x_1 + x_2$

$$\begin{aligned} \text{max} \quad & -7x_1' + 5x_2' \\ \text{s.t} \quad & x_1' + x_2' \geq 0 \\ & x_1' - x_2' \geq 200 \\ & 2x_1' \geq 100 \\ & x_1' + x_2' \leq 200 \\ & 100 + x_2' \geq x_1' \end{aligned}$$

Simplex method: example

Make a transformation $x_1 = 200 - x'_1 - x'_2$ and $x_2 = 100 - x'_1 + x'_2$

$$
\begin{aligned}
\text{max} \quad & -7x'_1 + 5x'_2 \\
\text{s.t} \quad & x'_1 + x'_2 \geq 0 \\
& x'_1 - x'_2 \geq 200 \\
& 2x'_1 \geq 100 \\
& x'_1 + x'_2 \leq 200 \\
& 100 + x'_2 \geq x'_1
\end{aligned}
$$

Simplex method: example

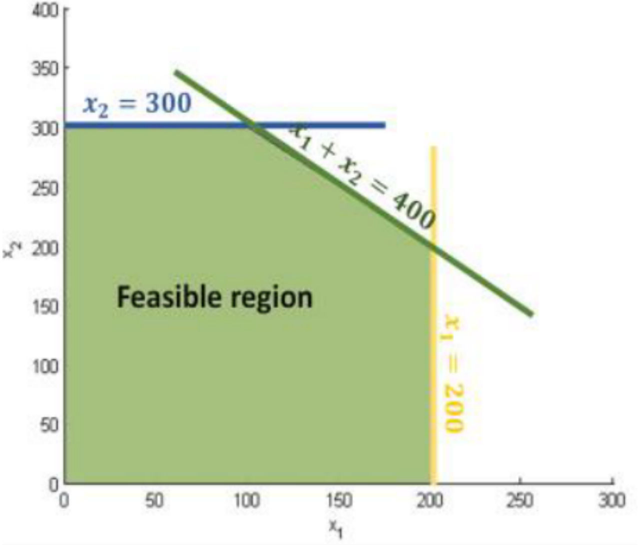
Make a transformation $x_1 = 200 - x_1' - x_2'$ and $x_2 = 100 - x_1' + x_2'$

$$\begin{aligned} \text{max} \quad & -7x_1' + 5x_2' \\ \text{s.t} \quad & x_1' + x_2' \geq 0 \\ & x_1' - x_2' \geq 200 \\ & 2x_1' \geq 100 \\ & x_1' + x_2' \leq 200 \\ & 100 + x_2' \geq x_1' \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

Simplex method: example

$$\begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ x_1 + & x_2 \leq 400 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

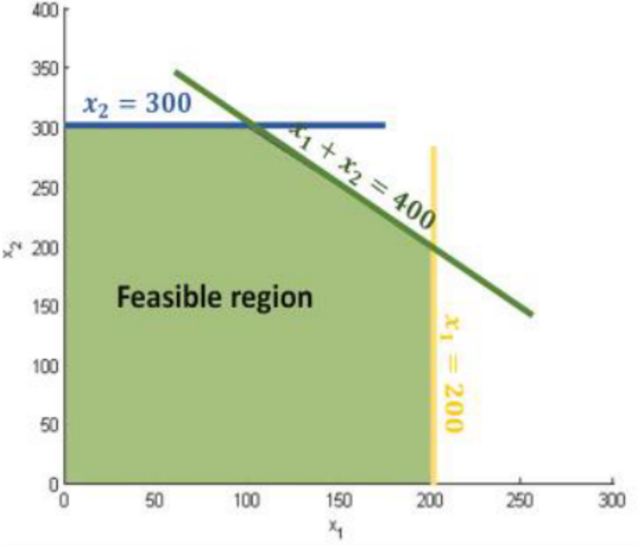
We start at the origin.



Simplex method: example

$$
\begin{aligned}
\text{max} \quad & x_1 + 6x_2 \\
\text{s.t} \quad & x_1 \leq 200 \\
& x_2 \leq 300 \\
x_1 + & x_2 \leq 400 \\
& x_1 \geq 0 \\
& x_2 \geq 0
\end{aligned}
$$

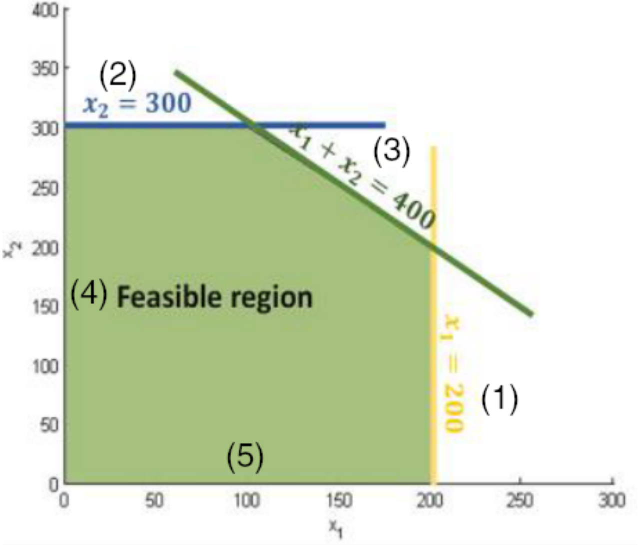
We start at the origin.



Simplex method: example

Iteration 1: c_i's are all positive. So we can increase either x_1 or x_2. Here, we increase x_1

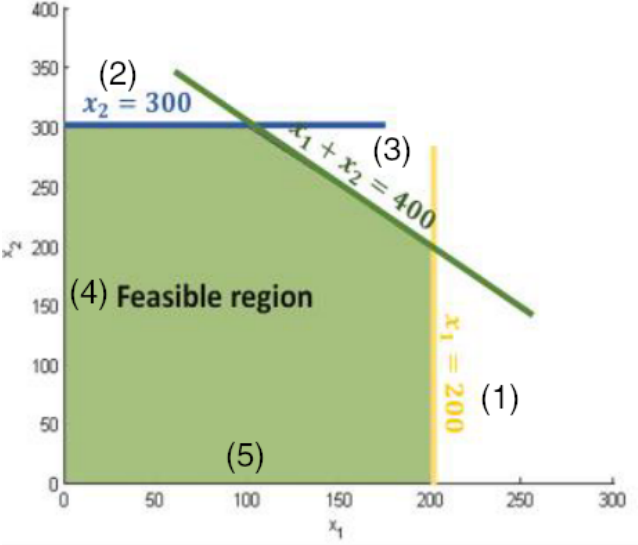
$$\begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 & (1) \\ & x_2 \leq 300 & (2) \\ x_1 + {} & x_2 \leq 400 & (3) \\ & x_1 \geq 0 & (4) \\ & x_2 \geq 0 & (5) \end{aligned}$$



Simplex method: example

Iteration 1: Move the point to the origin by transforming $y_1 = 200 - x_1$ and $y_2 = x_2$

$$
\begin{aligned}
\max \quad & x_1 + 6x_2 & \\
\text{s.t} \quad & x_1 \leq 200 & (1) \\
& x_2 \leq 300 & (2) \\
& x_1 + x_2 \leq 400 & (3) \\
& x_1 \geq 0 & (4) \\
& x_2 \geq 0 & (5)
\end{aligned}
$$

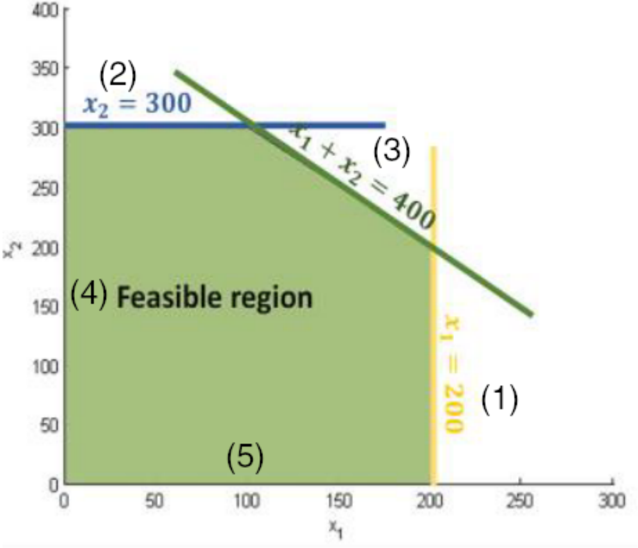


Simplex method: example

Iteration 1: Move the point to the origin by transforming $y_1 = 200 - x_1$ and $y_2 = x_2$

$$\max \; 200 - y_1 + 6y_2$$
$$\text{s.t} \quad y_1 \geq 0 \qquad (1)$$
$$y_2 \leq 300 \qquad (2)$$
$$-y_1 + y_2 \leq 200 \qquad (3)$$
$$y_1 \geq 200 \qquad (4)$$
$$y_2 \geq 0 \qquad (5)$$

Since one of c_i's is positive (6), we must continue

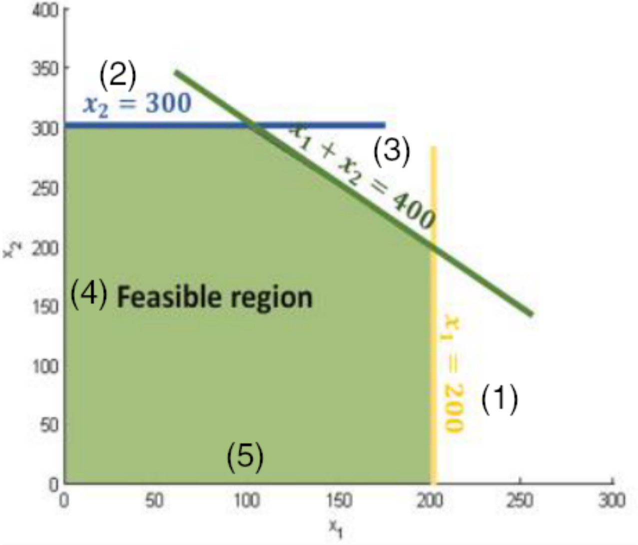


Simplex method: example

Iteration 2: Increase y_2 to leave (5) and continue on (1)

$$\max \; 200 - y_1 + 6y_2$$

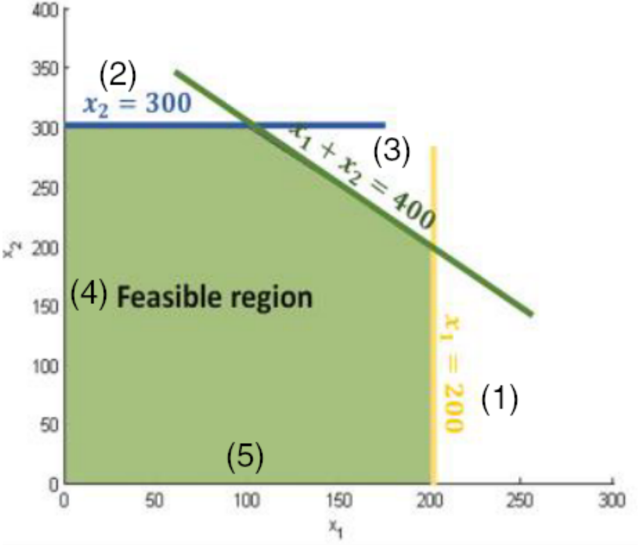
s.t	$y_1 \geq 0$	(1)
	$y_2 \leq 300$	(2)
	$-y_1 + y_2 \leq 200$	(3)
	$y_1 \geq 200$	(4)
	$y_2 \geq 0$	(5)



Simplex method: example

Iteration 2: Since we are at the intersection of (1) and (3), transform $z_1 = y_1$ and $z_2 = 200 + y_1 - y_2$

$$
\begin{aligned}
\text{max} \quad & 200 - y_1 + 6y_2 & \\
\text{s.t} \quad & y_1 \geq 0 & (1) \\
& y_2 \leq 300 & (2) \\
& -y_1 + y_2 \leq 200 & (3) \\
& y_1 \geq 200 & (4) \\
& y_2 \geq 0 & (5)
\end{aligned}
$$



Simplex method: example

Iteration 2: Since we are at the intersection of (1) and (3), transform $z_1 = y_1$ and $z_2 = 200 + y_1 - y_2$

$$\max \ 1400 + 5z_1 - 6z_2$$

$$\text{s.t} \ z_1 \geq 0 \quad (1)$$

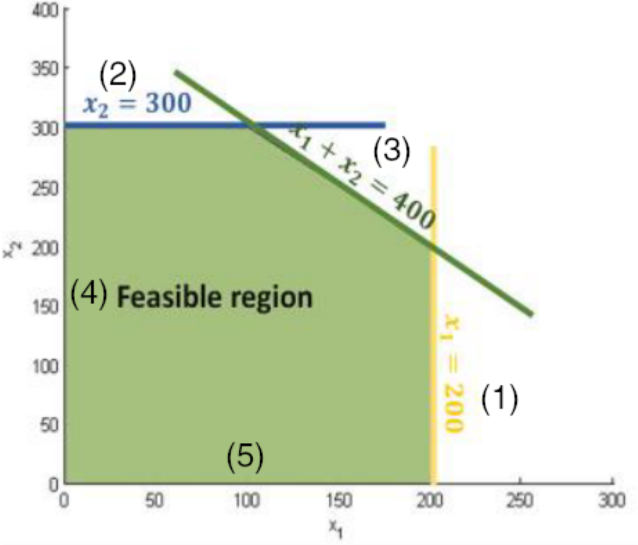
$$z_1 - z_2 \leq 100 \quad (2)$$

$$z_2 \geq 0 \quad (3)$$

$$z_1 \leq 200 \quad (4)$$

$$z_1 - z_2 \geq -200 \quad (5)$$

Since one of c_i's is positive (5), we must continue

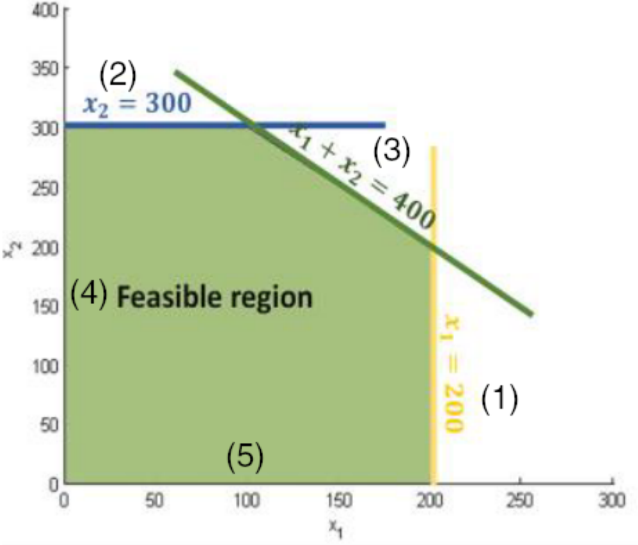


Simplex method: example

Iteration 3: Increase z_1 to leave (1) and continue on (3)

$$\max \ 1400 + 5z_1 - 6z_2$$

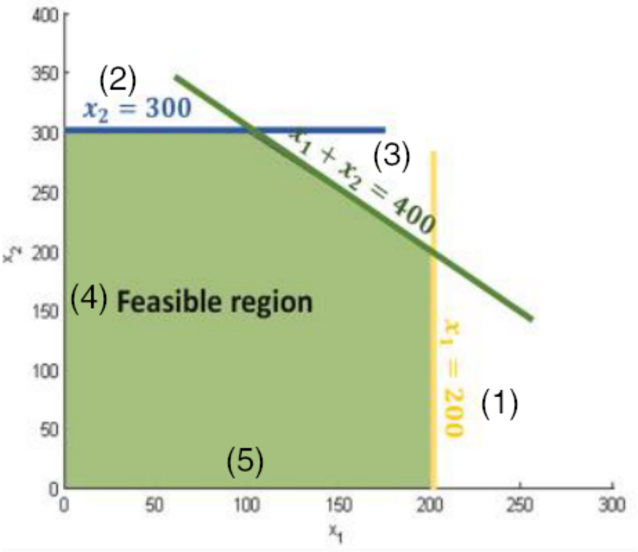
s.t	$z_1 \geq 0$	(1)
	$z_1 - z_2 \leq 100$	(2)
	$z_2 \geq 0$	(3)
	$z_1 \leq 200$	(4)
	$z_1 - z_2 \geq -200$	(5)



Simplex method: example

Iteration 3: Since we are at the intersection of (2) and (3), transform $w_1 = 100 - z_1 + z_2$ and $w_2 = z_2$

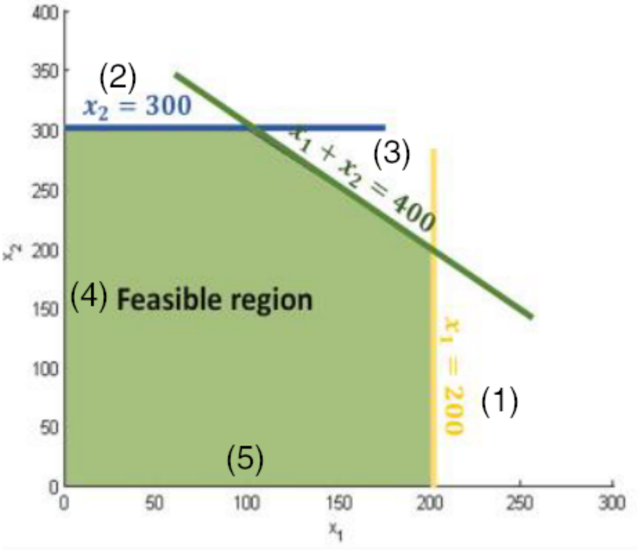
$$
\begin{aligned}
\text{max} \quad & 1400 + 5z_1 - 6z_2 \\
\text{s.t} \quad & z_1 \geq 0 && (1) \\
& z_1 - z_2 \leq 100 && (2) \\
& z_2 \geq 0 && (3) \\
& z_1 \leq 200 && (4) \\
& z_1 - z_2 \geq -200 && (5)
\end{aligned}
$$



Simplex method: example

Iteration 3: Since we are at the intersection of (2) and (3), transform $w_1 = 100 - z_1 + z_2$ and $w_2 = z_2$

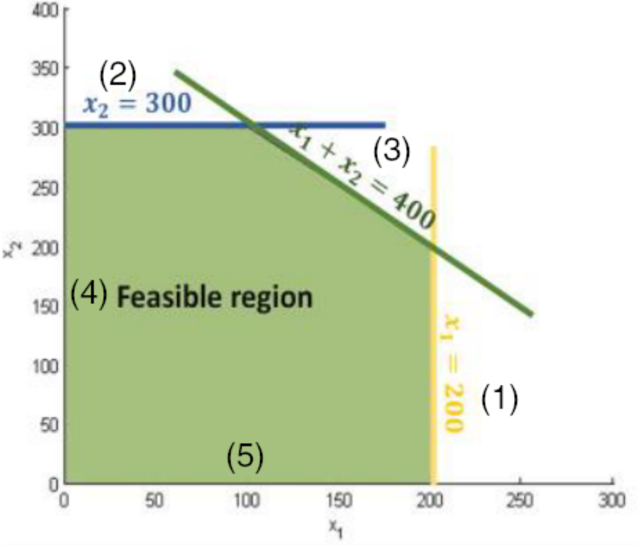
$$
\begin{aligned}
\text{max} \quad & 1900 - 5w_1 - w_2 & \\
\text{s.t} \quad & w_1 - w_2 \leq 0 & (1) \\
& w_1 \geq 0 & (2) \\
& w_2 \geq 0 & (3) \\
& w_1 - w_2 \geq -100 & (4) \\
& w_1 \leq 300 & (5)
\end{aligned}
$$



Simplex method: example

Since all coefficients are negative, this is maximized when $w_1 = w_2 = 0$, and the maximum is 1900

$$
\begin{aligned}
\text{max}\ \ & 1900 - 5w_1 - w_2 \\
\text{s.t}\ \ & w_1 - w_2 \leq 0 && (1) \\
& w_1 \geq 0 && (2) \\
& w_2 \geq 0 && (3) \\
& w_1 - w_2 \geq -100 && (4) \\
& w_1 \leq 300 && (5)
\end{aligned}
$$



Simplex method: an example

Duality

The idea behind duality

- The feasible and optimal solutions of the dual provide very useful information about the original (**primal**) LP

- In particular, if the primal LP is a maximization problem, the dual can be used to find upper bounds on its optimal value. (Similarly, if the primal is a minimization problem, the dual gives lower bounds)

Example

$$\begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 && (1) \\ & x_2 \leq 300 && (2) \\ x_1 + {}& x_2 \leq 400 && (3) \\ x_1, {}& x_2 \geq 0 \end{aligned}$$

- Somebody comes to you and claims $x_1 = 100, x_2 = 300$ with optimal value 1900. How can we check this claim?

Example

$$
\begin{aligned}
\text{max} \quad & x_1 + 6x_2 & \\
\text{s.t} \quad & x_1 \leq 200 & (1) \\
& x_2 \leq 300 & (2) \\
x_1 + {} & x_2 \leq 400 & (3) \\
x_1, {} & x_2 \geq 0 &
\end{aligned}
$$

- Somebody comes to you and claims $x_1 = 100, x_2 = 300$ with optimal value 1900. How can we check this claim?
- Well, we can combine the constraints to produce an upper bound of the objective function. For example,

$$(1) + 6(2) \implies x_1 + 6x_2 \leq 2000$$

Example

$$
\begin{aligned}
\text{max} \quad & x_1 + 6x_2 \\
\text{s.t} \quad & x_1 \leq 200 && (1) \\
& x_2 \leq 300 && (2) \\
x_1 + & x_2 \leq 400 && (3) \\
x_1, & x_2 \geq 0
\end{aligned}
$$

- Somebody comes to you and claims $x_1 = 100, x_2 = 300$ with optimal value 1900. How can we check this claim?
- Well, we can combine the constraints to produce an upper bound of the objective function. For example,

$$(1) + 6(2) \implies x_1 + 6x_2 \leq 2000$$

- Can we bring down this upper bound further?

Example

$$\begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 && (1) \\ & x_2 \leq 300 && (2) \\ x_1 + {} & x_2 \leq 400 && (3) \\ x_1, {} & x_2 \geq 0 \end{aligned}$$

- What if we try

$$0(1) + 5(2) + 1(3) \implies x_1 + 6x_2 \leq 1900$$

Example

$$\begin{aligned} \text{max} \quad x_1 + 6x_2 & \\ \text{s.t} \quad x_1 \le 200 & \qquad (1) \\ x_2 \le 300 & \qquad (2) \\ x_1 + x_2 \le 400 & \qquad (3) \\ x_1, x_2 \ge 0 & \end{aligned}$$

- What if we try

$$0(1) + 5(2) + 1(3) \implies x_1 + 6x_2 \le 1900$$

- This shows that $x_1 = 100, x_2 = 300$ is optimal
- The coefficients $(0, 5, 1)$ are called **dual multiplier**

Dual form

Let's formalize what we just did. Start by introducing multiplier for each constraint:

$$
\begin{aligned}
x_1 &\leq 200 \qquad y_1 & (1)\\
x_2 &\leq 300 \qquad y_2 & (2)\\
x_1 + x_2 &\leq 400 \qquad y_3 & (3)
\end{aligned}
$$

Dual form

Let’s formalize what we just did. Start by introducing multiplier for each constraint:

$$
\begin{aligned}
x_1 &\leq 200 \qquad && y_1 && (1)\\
x_2 &\leq 300 \qquad && y_2 && (2)\\
x_1 + x_2 &\leq 400 \qquad && y_3 && (3)
\end{aligned}
$$

- First, we need $y_1, y_2, y_3 \geq 0$ to preserve the inequalities after multiplication

Dual form

Let's formalize what we just did. Start by introducing multiplier for each constraint:

$$x_1 \leq 200 \qquad y_1 \tag{1}$$

$$x_2 \leq 300 \qquad y_2 \tag{2}$$

$$x_1 + x_2 \leq 400 \qquad y_3 \tag{3}$$

- Then we multiply the equations with y_i and add them up

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

Dual form

Let's formalize what we just did. Start by introducing multiplier for each constraint:

$$x_1 \leq 200 \qquad y_1 \tag{1}$$

$$x_2 \leq 300 \qquad y_2 \tag{2}$$

$$x_1 + x_2 \leq 400 \qquad y_3 \tag{3}$$

- Then we multiply the equations with y_i and add them up

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

  We need the left hand side to be the objective function. This can be achieved by enforcing

$$y_1 + y_3 \geq 1$$

$$y_2 + y_3 \geq 6$$

Dual form

Let's formalize what we just did. Start by introducing multiplier for each constraint:

$$x_1 \leq 200 \qquad y_1 \tag{1}$$
$$x_2 \leq 300 \qquad y_2 \tag{2}$$
$$x_1 + x_2 \leq 400 \qquad y_3 \tag{3}$$

- Then we multiply the equations with y_i and add them up

  $$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

  Finally, we want to get the best possible upper bound which means that we want to minimize $200y_1 + 300y_2 + 400y_3$

Dual form

Altogether, this gives us the following linear programming problem

$$\begin{aligned} \text{min} \;\; 200y_1 &+ 300y_2 + 400y_3 \\ \text{s.t} \;\; x_1 &\leq 200 \\ y_1 + y_3 &\geq 1 \\ y_2 + y_3 &\geq 6 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

This problem is called the **dual LP**

Primal and Dual LP

$$\begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The optimal solution is

$$x_1 = 100, x_2 = 300$$

with optimal value 1900

$$\begin{aligned} \text{min} \quad & 200y_1 + 300y_2 + 400y_3 \\ \text{s.t} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

The optimal solution is

$$y_1 = 0, y_2 = 5, y_3 = 1$$

with optimal value 1900

Dual form

More generally, if the primal LP (P) is of the form

$$\max\ c^T x$$
$$\text{s.t}\ Ax \leq b$$
$$x \geq 0$$

Then the dual LP (D) will be of the form

$$\min\ b^T y$$
$$\text{s.t}\ A^T y \geq c$$
$$y \geq 0$$

Weak duality and strong duality

Here are two fundamental theorems of duality:

Weak duality. If x is feasible for (P) and y is feasible for (D), then $c^T x \leq b^T y$

Weak duality and strong duality

Here are two fundamental theorems of duality:

Weak duality. If x is feasible for (P) and y is feasible for (D), then $c^T x \leq b^T y$

Strong duality: If (P) has an optimal value, then so does (D) and the two optimal values coincide

Weak duality

Weak duality. If x is feasible for (P) and y is feasible for (D), then $c^T x \leq b^T y$

$$\begin{aligned} \text{max} \quad & x_1 + 6x_2 \\ \text{s.t} \quad & x_1 \leq 200 \\ & x_2 \leq 300 \\ & x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{min} \quad & 200y_1 + 300y_2 + 400y_3 \\ \text{s.t} \quad & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Weak duality

Weak duality. If x is feasible for (P) and y is feasible for (D), then $c^T x \leq b^T y$

Other forms of Primal/Dual pair

The Primal/Dual pair can appear in other forms. For example,

$$\begin{aligned} \text{min} \;\; & c^T x \\ \text{s.t} \;\; & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \text{max} \;\; & b^T y \\ \text{s.t} \;\; & A^T y \leq c \end{aligned} \qquad \text{(D')}$$

Other forms of Primal/Dual pair

The Primal/Dual pair can appear in other forms. For example,

$$\begin{aligned} \min\ & c^T x \\ \text{s.t}\ & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{(P')}$$

$$\begin{aligned} \max\ & b^T y \\ \text{s.t}\ & A^T y \leq c \end{aligned} \qquad \text{(D')}$$

Weak duality. If x is feasible for (P') and y is feasible for (D'), then $c^T x \geq b^T y$

General form of Primal/Dual pair

Here's the general form of the Primal/Dual pair:

Let I, E, N be some subsets of $\{1, 2, \ldots, m\}$

$$
\begin{aligned}
\text{max} \quad & c_1 x_1 + \ldots + c_n x_n \\
\text{s.t} \quad & a_{i1} x_1 + \ldots + a_{in} x_n \leq b_i \quad \text{for } i \in I \\
& a_{i1} x_1 + \ldots + a_{in} x_n = b_i \quad \text{for } i \in E \\
& x_i \geq 0 \quad \text{for } i \in N
\end{aligned}
$$

$$
\begin{aligned}
\text{min} \quad & b_1 y_1 + \ldots + b_m y_m \\
\text{s.t} \quad & a_{1j} y_1 + \ldots + a_{mj} y_m \geq c_j \quad \text{for } j \in N \\
& a_{1j} y_1 + \ldots + a_{mj} y_m = c_j \quad \text{for } j \notin N \\
& y_j \geq 0 \quad \text{for } j \in I
\end{aligned}
$$

Primal/dual possibilities

We go back to the following forms of the primal and dual

$$\begin{aligned} \max \ & c^T x \\ \text{s.t } & Ax \leq b \\ & x \geq 0 \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min \ & b^T y \\ \text{s.t } & A^T y \geq c \\ & y \geq 0 \end{aligned} \qquad \text{(D)}$$

Primal/dual possibilities

We go back to the following forms of the primal and dual

$$\begin{aligned} \text{max } & c^T x \\ \text{s.t } & Ax \leq b \\ & x \geq 0 \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \text{min } & b^T y \\ \text{s.t } & A^T y \geq c \\ & y \geq 0 \end{aligned} \qquad \text{(D)}$$

We know that an LP can have three possibilities: either it has a solution, or it is unbounded, or it is infeasible. Here are the possibilities that we can have when we consider a primal/dual pair:

		PRIMAL		
		Finite optimal	Unbounded	Infeasible
DUAL	Finite optimal	Possible	Impossible	Impossible
	Unbounded	Impossible	Impossible	Possible
	Infeasible	Impossible	Possible	Possible