Game theory

Source: Ahmadi, ORF 363 slides.

Zero-sum game

- Two people make decisions at the same time
- The payoff depends on the joint decisions
- Whatever one person wins the other person loses

Rock-Paper-Scissors

Payoffs are from **row** player to **column** player:

		Player B		
		Rock	Paper	Scissors
Player A	Rock	0	1	-1
	Paper	1	0	-1
	Scissors	-1	1	0

Any deterministic strategy by either player can be defeated systematically by the other player

Two-Player Zero-Sum games

Given: $m \times n$ matrix $A = \{a_{ij}\}$

- Row player selects a strategy $i \in \{1, ..., m\}$
- Column player selects a strategy $j \in \{1, ..., n\}$
- Row player pays column player a_{ij} dollars

Deterministic strategies can be (and usually are) bad

Randomized Strategies

- Suppose row player pick i with probability y_i
  - We say that row player uses **strategy** $y = (y_1, y_2, \ldots, y_m)$

- Suppose column player pick j with probability x_j
  - We say that column player uses **strategy** $x = (x_1, x_2, \ldots, x_n)$

	x_1	x_2	x_3
y_1	a_{11}	a_{12}	a_{13}
y_2	a_{21}	a_{22}	a_{23}
y_3	a_{31}	a_{32}	a_{33}

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Expected payoff that row player has to pay to column player is

$$\sum_i \sum_j y_i a_{ij} x_j = y^T A x$$

Column Player's Analysis

Suppose column player were to adopt strategy x

Then, row player's best defense is to use strategy y that minimizes the expected payoff

$$\min_y \; y^T A x$$

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And so column player should choose x which maximizes these possibilities

$$\max_{x} \min_{y} \; y^T A x$$

Quiz

What's the solution to this problem?

$$
\begin{aligned}
\min\ & 3y_1 + 6y_2 + 2y_3 + 18y_4 - 7y_5 \\
\text{s.t. } & y_1 + y_2 + y_3 + y_4 + y_5 = 1 \\
& y_i \geq 0, i = 1, 2, 3, 4, 5
\end{aligned}
$$

Solving Max-Min Problems as LPs

Define

$$e_i = (0, 0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{-th position}}}{1}, 0, \ldots, 0)$$

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Column player should choose x that solves:

$$\max_x \min_i \; e_i^T A x$$

$$\text{s.t.} \; \sum_j x_j = 1$$

$$x_j \geq 0, j = 1, 2, \ldots n$$

Reduction to an LP

$$\max_x \min_i \; e_i^T A x$$

$$\text{s.t. } \sum_j x_j = 1$$

$$x_j \geq 0, j = 1, 2, \ldots n$$

Let x be a solution and $v = \min_i \; e_i^T A x$

In matrix form

$$\max_{x,v} v$$

s.t. $ve - Ax \leq 0$

$$e^T x = 1$$

$$x \geq 0$$

Row Player's Perspective

Similarly, row player seeks y^* attaining:

$$\min_y \max_x \; y^T A x =$$

which is equivalent to:

Duality

Row player's problem is dual to column player's:

Column player

$$\max_{x,v} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ v \end{bmatrix}$$

s.t. $\begin{bmatrix} -A & e \end{bmatrix} x \leq 0$

$\begin{bmatrix} e^T & 0 \end{bmatrix} v = 1$

$x \geq 0$

Row player

$$\min_{y,u} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} y \\ u \end{bmatrix}$$

s.t. $\begin{bmatrix} -A^T & e \end{bmatrix} y \geq 0$

$\begin{bmatrix} e^T & 0 \end{bmatrix} u = 1$

$y \geq 0$

Minimax Theorem

Theorem. Let x^* denote column player's solution to her max-min problem
Let y^* denote row player's solution to his min-max problem
Then

$$\max_x \; y^{*T} A x = \min_y y^T A x^*$$

Proof. From Strong Duality Theorem, we have

Example: Ultra-Conservative Investor

An investor is trying to decide what combination of actions to take among three possible courses of action

Rates of return are shown in the following table

		States of Nature			
		Growth	Medium G	No Change	Low
Actions	Bonds	12%	15	7	3
	Stocks	15	9	5	-2
	Deposit	7	7	7	7

This investment problem can be formulated as if the investor is playing a game against nature

LP formulation

		States of Nature			
		Growth	Medium G	No Change	Low
Actions	Bonds	12%	15	7	3
	Stocks	15	9	5	-2
	Deposit	7	7	7	7

Suppose our investor has $\$100,000$ to allocate among the three possible investments with the unknown amounts y_1, y_2, y_3, respectively, that is,

$$y_1 + y_2 + y_3 = 100,000$$

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Under these conditions, the returns are:

$$0.12y_1 + 0.15y_2 + 0.07y_3 \quad \text{(if Growth (G))}$$

$$0.08y_1 + 0.09y_2 + 0.07y_3 \quad \text{(if Medium G)}$$

$$0.07y_1 + 0.05y_2 + 0.07y_3 \quad \text{(if No Change)}$$

$$0.03y_1 - 0.02y_2 + 0.07y_3 \quad \text{(if Low)}$$

LP formulation

The objective is that the smallest return v is as large as possible

The corresponding LP is:

$$\max_{y_1,y_2,y_3,v} \quad v$$

$$\text{s.t.} \quad : y_1 + y_2 + y_3 = 100,000$$

$$0.12y_1 + 0.15y_2 + 0.07y_3 \geq v$$

$$0.08y_1 + 0.09y_2 + 0.07y_3 \geq v$$

$$0.07y_1 + 0.05y_2 + 0.07y_3 \geq v$$

$$0.03y_1 - 0.02y_2 + 0.07y_3 \geq v$$

$$y_1, y_2, y_3 \geq 0$$

This LP formulation is similar to the max-min problem

Example: Ultra-Conservative Investor

$$\max_{y_1, y_2, y_3, v} \quad v$$

$$\text{s.t.} \quad : y_1 + y_2 + y_3 = 100,000$$

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$$0.03y_1 - 0.02y_2 + 0.07y_3 \geq v$$

$$y_1, y_2, y_3 \geq 0$$

- Solving this problem with an LP solution algorithm, the optimal solution is $y_1 = 0, y_2 = 0, y_3 = \$100,000$, and $v = \$7000$
- the investor must put all the money in the money market account with the accumulated return of $100,000 \times 1.07 = \$10,7000$

Example: Commuting

You are on your morning commute, and you have three choices of how to get to work

Suppose that for every day, one of four possible scenarios occur

	Good day	Bad for highway	Bad for roads#1	Bad for roads# 2
Highway	30	80	30	30
Roads#1	40	40	90	40
Road#2	50	50	50	75

Assume that the worst scenario always occurs. What is your best strategy?

Example: Commuting

	Good day	Bad for highway	Bad for roads#1	Bad for roads# 2
Highway	30	80	30	30
Roads#1	40	40	90	40
Road#2	50	50	50	75

Pure strategy might not be the best if you don’t want tp see the worst case scenario

So let’s try a mixed strategy

Example: Commuting

	Good day	Bad for highway	Bad for roads#1	Bad for roads# 2
Highway	30	80	30	30
Roads#1	40	40	90	40
Road#2	50	50	50	75

The corresponding LP is:

$$\min_{y_1,y_2,y_3,u} \quad u$$

s.t. $: y_1 + y_2 + y_3 = 1$

$$30y_1 + 40y_2 + 50y_3 \leq u$$

$$80y_1 + 40y_2 + 50y_3 \leq u$$

$$30y_1 + 90y_2 + 50y_3 \leq u$$

$$30y_1 + 40y_2 + 75y_3 \leq u$$

$$y_1, y_2, y_3 \geq 0$$

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The corresponding LP is:

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$$30y_1 + 40y_2 + 50y_3 \leq u$$

$$80y_1 + 40y_2 + 50y_3 \leq u$$

$$30y_1 + 90y_2 + 50y_3 \leq u$$

$$30y_1 + 40y_2 + 75y_3 \leq u$$

$$y_1, y_2, y_3 \geq 0$$

The solution to this LP is:

$$y_1 = 0.25, y_2 = 0.25, y_3 = 0.5$$

and the minimum value of u is

$$u = 55$$