Online convex optimization

Source: David Childer's 73-423 note

Online learning

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  - Web services: may need to update page every day or even every visitor
  - Large retail operation: need to predict inventories for each outlet each day

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- These goals motivate **online learning** methods
  - A class of algorithms for making a sequence of repeated forecasting decisions

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What do we want to optimize?

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- Let M_* be the **best** model if we were to observe $(x_1, y_1), \ldots, (x_T, y_T)$ from the beginning
- Then, M_* minimizes the **total loss**

$$\sum_{t=1}^{T} \ell(M_*(x_t), y_t)$$

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The performance is measured in total loss, so we want to minimize the so-called **regret**:

$$\text{Regret}_T = \max_{y_1,\ldots,y_T} \underbrace{\sum_{t=1}^{T} \ell(M_t(x_t), y_t)}_{\text{loss from online learning}} - \underbrace{\sum_{t=1}^{T} \ell(M_*(x_t), y_t)}_{\text{loss from the best model}}$$

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Of course Regret_T grows as a function of T but how slow does it grow?

For example, $\text{Regret}_T \approx \sqrt{T}$ is better than $\text{Regret}_T \approx T$

Example: online classification

Suppose that we have a stream of data $x_1, y_1, x_2, y_2, \ldots, x_T, y_T$, where $x_t \in \mathbb{R}^d$ and $y_t \in \{-1, 1\}$

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We will consider the $0-1$ loss:

$$\ell(M(x_t), y_t) = \begin{cases} 0 & \text{if } M(y_t) = y_t \\ 1 & \text{if } M(y_t) \neq y_t \end{cases}$$

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We will use a linear model: $M_t(x_t) = \text{sign}(\theta_t^T x_t)$. Here, θ_t is the vector of coefficients at time t

Perceptron Algorithm (Rosenblatt, 1958)

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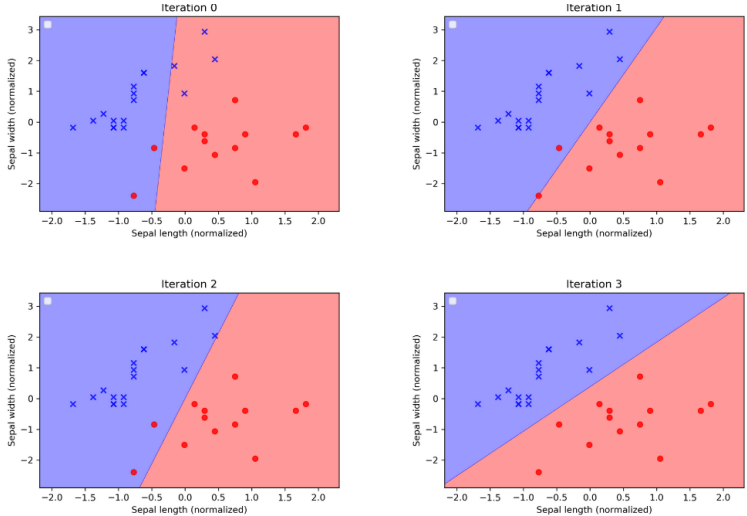
4. $\theta_{t+1} = \theta_t + \ell_t y_t x_t$

end

Example: Iris dataset



Source: Simone Alberto Peirone (2019)

Follow the leader

Suppose that we have a stream of data $x_1, y_1, x_2, y_2, \ldots, x_T, y_T$, where $x_t \in \mathbb{R}^d$ and $y_t \in \mathbb{R}$

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Why not pick coefficients θ_t which have performed best so far?

- After step t, find θ_t that minimizes the total loss:

$$\sum_{s=1}^{t-1} \ell(M_{\theta_t}(x_s), y_s)$$

This method of choosing θ_t is called **follow the leader (FTL)**

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- So does it work? Can we just do what we were already doing and get good results in the online setting?
  - In general, **no**
  - For general convex loss functions, regret can grow linearly in T

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Then, at step t,

$$\sum_{s=1}^{t-1} \ell_{\theta_t}(x_s) = (-0.5 + 1 - 1 + \ldots)\theta_t = \begin{cases} -0.5\theta_t & t \text{ even} \\ 0.5\theta_t & t \text{ odd} \end{cases}$$

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So the loss is

$$\sum_{t=1}^{T} \ell_{\theta_t}(x_t) = \begin{cases} -0.5\theta_1 + T - 1 & t \text{ even} \\ -0.5\theta_1 + T - 1 & t \text{ odd} \end{cases}$$

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But the best choice of θ is 0, since then we would have

$$\sum_{s=1}^{t} \ell_\theta(x_t) = (-0.5 + 1 - 1 + \ldots)\theta = 0$$

Thus, our regret is $(-0.5\theta_1 + T - 1) - 0 = O(T)$