Online convex optimization 2

Source: David Childer’s 73-423 note

Review

- Last time, we talked about two online learning algorithm: Perceptron and Follow the Leader (FTL)

- We showed that FTL can have a worst-case regret of order T. We now introduce a new algorithm that, under some conditions, can produce regret of order $\sqrt{T}$

Follow the Regularized Leader (FTRL)

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At step t:

$$\text{Choose } \theta_t \text{ that minimizes } \sum_{s=1}^{t-1} \ell_s(\theta_t) + \lambda R(\theta_t)$$

where R is a **regularizer**

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- Square regularizer: $R(x) = \frac{1}{2}\|x\|_2^2$
- Entropic regularizer: $R(x) = \sum_{i=1}^{d} x_i \log x_i$ over $\{x \in \mathbb{R}^d \ : \ \sum_i x_i = 1, x_i \geq 0\}$

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Parameter $\lambda > 0$ determines strength of the regularization: small values closer to FTL, large values closer to minimizing R

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Corresponds exactly to running regularized optimization each round

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- Solving the first-order condition for θ_t,

$$\theta_t = -\frac{1}{\lambda}\sum_{s=1}^{t-1} x_s = \theta_{t-1} - \frac{1}{\lambda}x_{t-1}$$

Algorithm: Online Linear Optimization

Input: A stream of data $x_1, x_2, , \ldots, x_T$, where $x_t \in \mathbb{R}^d$, a regularized parameter $\lambda > 0$

Initialize the coefficients $\theta_1 \in \mathbb{R}^d$

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With $\lambda \approx \sqrt{T}$ gives $\mathrm{Regret}_T = \sqrt{T}$

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- Taking the sum over t,

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- Define $\tilde{\ell}_t(\theta) = \nabla \ell_t(\theta_t)^T \theta$. We can rewrite the right-hand side

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- **Linearization trick**: replace ℓ_t with $\tilde{\ell}_t$ in the FTRL algorithm

Online Gradient Descent

- We will use the square regularizer:

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Online Gradient Descent

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for $t = 2$ **to** T **do**

1. Update $\theta_t = \theta_{t-1} - \frac{1}{\lambda}\nabla \ell_{t-1}(\theta_{t-1})$
2. Receive new data point
3. Compute loss gradient $\nabla \ell_t(\theta_t)$

end

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  - Set the first horizon T_1

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  - Set the first horizon T_1

    For $t = 1, \ldots, T_1$, run FTRL with $\lambda = \sqrt{T_1}$
  - When we reach T_k, set $T_{k+1} = 2T_k$

    For $t = T_k + 1, T_k + 2, \ldots, T_{k+1}$, run FTRL with $\lambda = \sqrt{T_k}$

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- For online gradient descent **with constraint**, we can enforce our update to be inside the feasible set
  - $\theta_t = \theta_{t-1} - \frac{1}{\lambda} \nabla \ell_{t-1}(\theta_{t-1})$
  - Move θ_t to the closest point in $\mathcal{K}$

This is called **projected online gradient descent**

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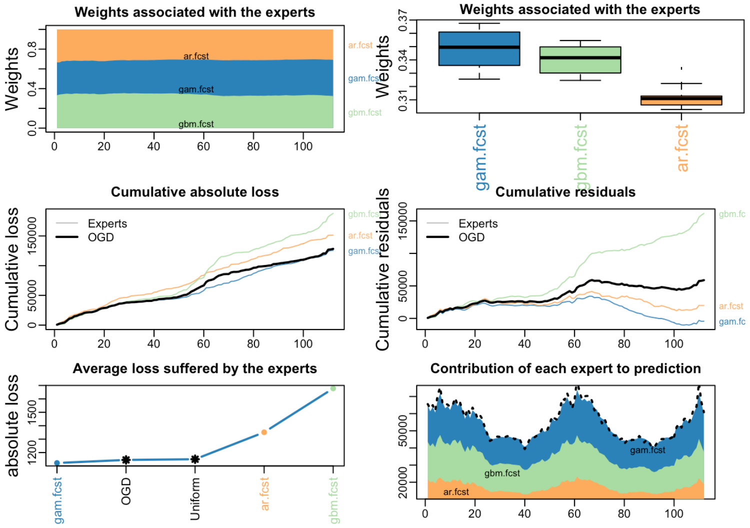
- Let $\lambda = \sqrt{t}$ for online gradient descent

Results


Weights associated with the experts
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OGD
Cumulative residuals
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Average loss suffered by the experts
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Uniform
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Contribution of each expert to prediction
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- Want to use ad and user-level attributes for prediction

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- Apply features in online logistic regression:

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- Uses regularizer depending on whole past sequence of θ_s, plus LASSO penalty
- Latter acts like Lasso penalty; former like square penality, but leads to more computationally efficient updates