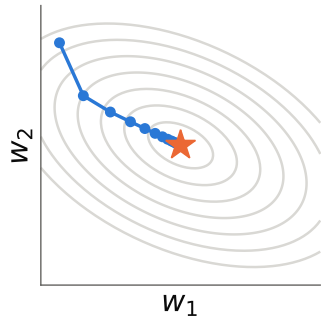
Linear optimization

208424 Lecture 8

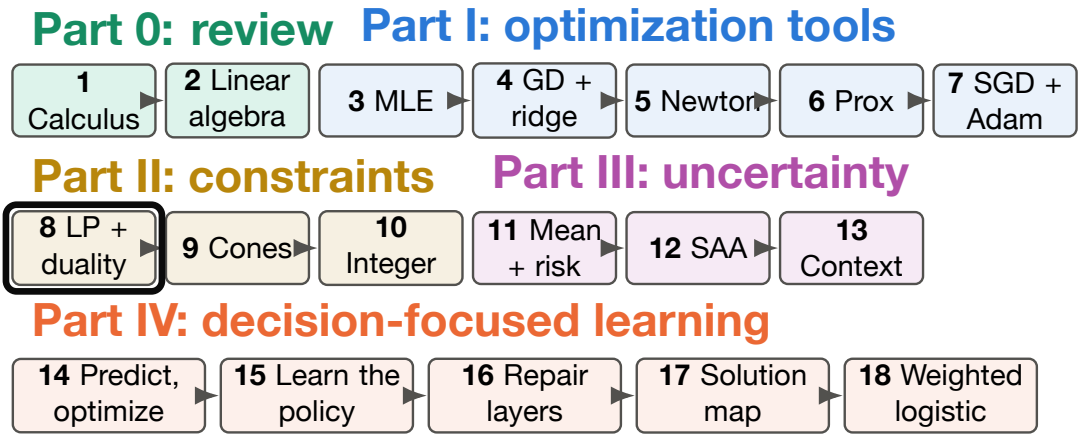
Lectures 3 to 7



We chose w to minimize a loss.

Any w was allowed.

Course map


Part 0: review
Part I: optimization tools
1 Calculus
2 Linear algebra
3 MLE
4 GD + ridge
5 Newton
6 Prox
7 SGD + Adam
Part II: constraints
Part III: uncertainty
8 LP + duality
9 Cones
10 Integer
11 Mean + risk
12 SAA
13 Context
Part IV: decision-focused learning
14 Predict, optimize
15 Learn the policy
16 Repair layers
17 Solution map
18 Weighted logistic

Lecture 8: Linear optimization



Photo: Jeremy Keith, CC BY 2.0, Wikimedia Commons

How many croissants and sandwiches each day?

The oven and the staff have 240 minutes a day.

Two products



croissant



sandwich

Each croissant gives 50 baht of profit.

Each sandwich gives 40 baht.

The decision variables





x_1 = croissants per day x_2 = sandwiches per day

x_1 and x_2 are the decision variables.

A plan is one choice of x_1 and x_2.

The objective

$$\text{Profit} = 50x_1 + 40x_2$$

We want the maximum profit.

The profit is our objective.

The oven



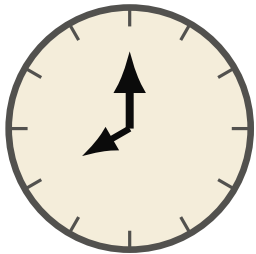
Photo: Jeremy Keith, CC BY 2.0, Wikimedia Commons

A croissant needs 2 minutes in the oven.

A sandwich needs 1 minute in the oven.

$$2x_1 + x_2 \leq 240$$

The staff



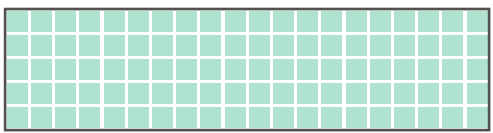
240 minutes a day

A croissant needs 1 minute of staff time.

A sandwich needs 2 minutes of staff time.

$$x_1 + 2x_2 \leq 240$$

The display



100 places

The display holds 100 sandwiches.

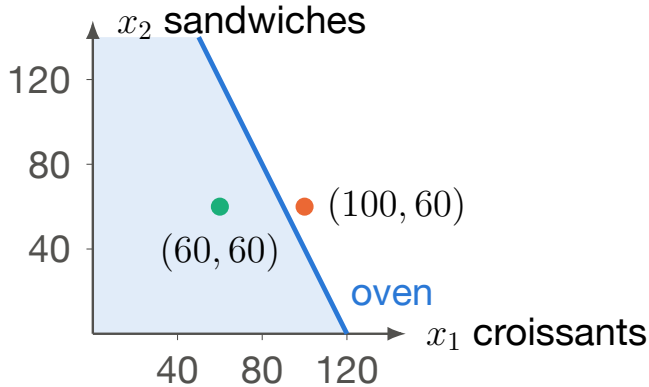
$$x_2 \leq 100$$

Constraints

$2x_1 + x_2 \leq 240$	$x_1 + 2x_2 \leq 240$	$x_2 \leq 100$	$x_1, x_2 \geq 0$
oven	staff	display	not negative

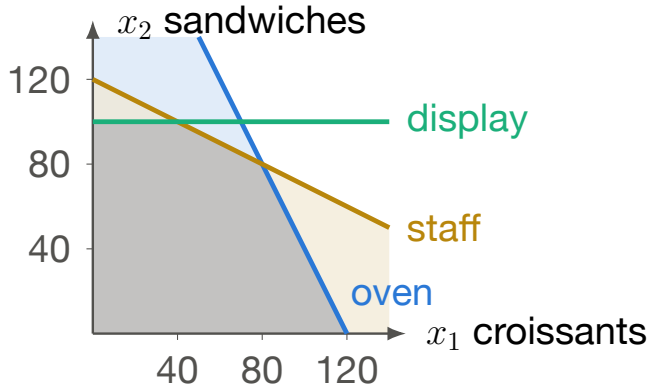
A constraint is a condition every plan must satisfy.

The oven constraint on a graph



Plans below the line satisfy the oven constraint.

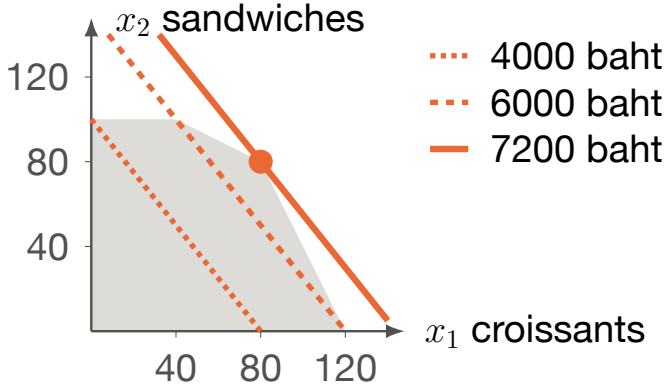
The feasible region



A feasible plan satisfies every constraint.

The feasible region is the set of feasible plans.

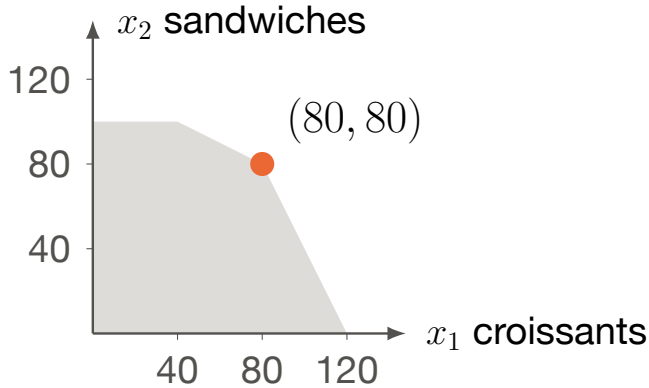
Lines of equal profit



Plans on one line earn the same profit.

Move the line up to the last point in the region.

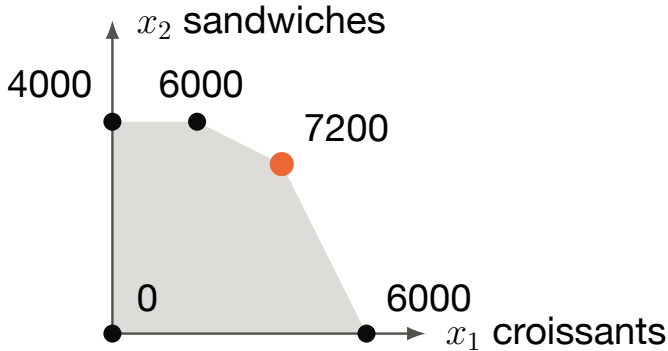
The optimal solution



The best plan $(80, 80)$ is the optimal solution.

The profit is $50 \times 80 + 40 \times 80 = 7200$ baht.

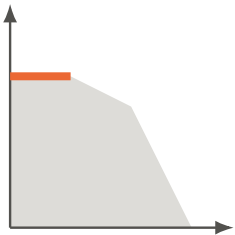
Vertices and their profits

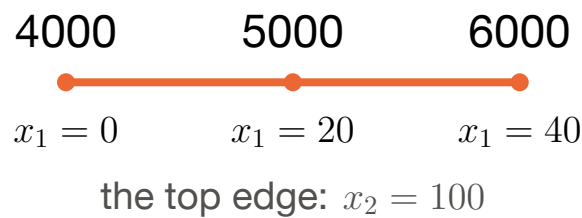


A vertex is a corner point of the region.

The best vertex is the best plan.

Why a vertex is optimal

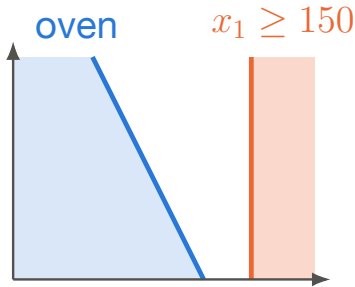




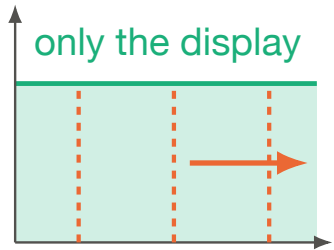
On an edge, the profit only rises or only falls.

So the maximum on an edge is at a vertex.

No best plan



infeasible



unbounded

Infeasible means no plan satisfies every constraint.

Unbounded means the profit has no maximum.

A linear optimization problem (LP)

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad c = \begin{pmatrix} 50 \\ 40 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 240 \\ 240 \\ 100 \end{pmatrix}$$

maximize $c^\top x$ subject to $Ax \leq b$, $x \geq 0$

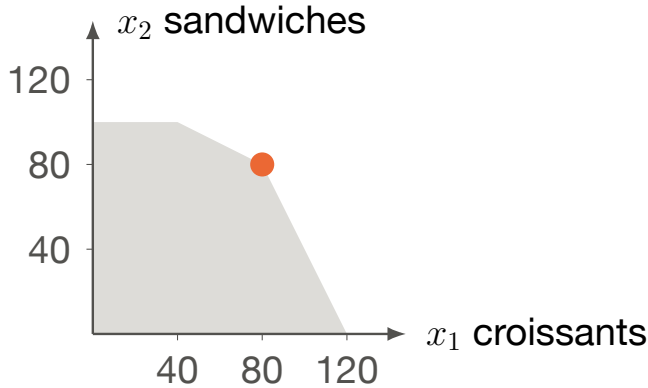
“Subject to” means every plan must satisfy these.

What linear means

Each term is a number multiplied by a variable.

$50x_1 + 40x_2$ is linear. x_1^2 and x_1x_2 are not.

Summary: linear optimization



An LP has a linear objective and linear constraints.

If an optimal solution exists, there is one at a vertex.

Checkpoint 1

Is the plan $(90, 80)$ feasible?

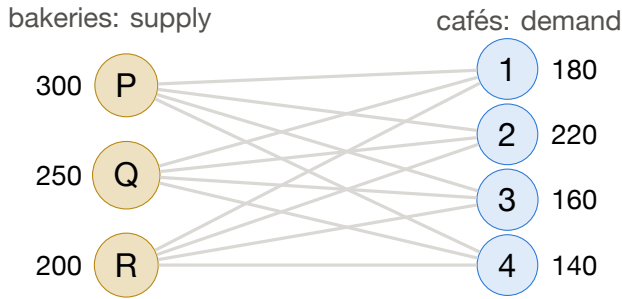
Checkpoint 1

Is the plan $(90, 80)$ feasible?

No. The oven needs $2 \times 90 + 80 = 260$ minutes.

The oven has only 240 minutes.

Larger problems



Real problems have many variables.

A solver is a program that finds the best plan.

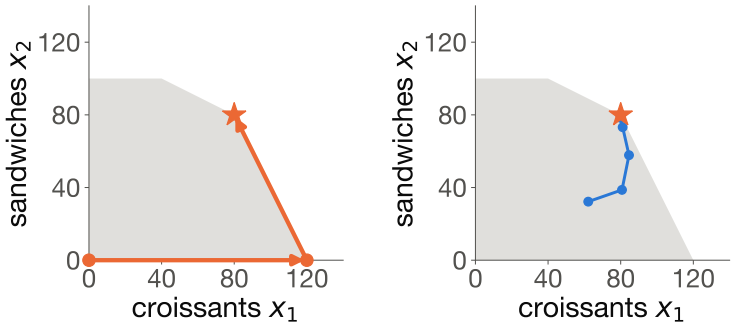
The bakery in Python

```
from scipy.optimize import linprog
A = [[2, 1], [1, 2], [0, 1]]
b = [240, 240, 100]
res = linprog([-50, -40], A_ub=A, b_ub=b)
res.x      # [80, 80]
-res.fun   # 7200
```

linprog minimizes, so we give it $-$profit.

linprog assumes $x \geq 0$.

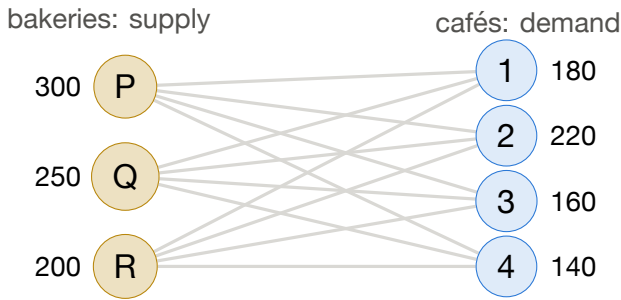
How a solver searches



The simplex method moves from vertex to vertex.

An interior-point method moves inside the region.

Shipping croissants



The bakeries must meet each café's demand.

Find the minimum delivery cost.

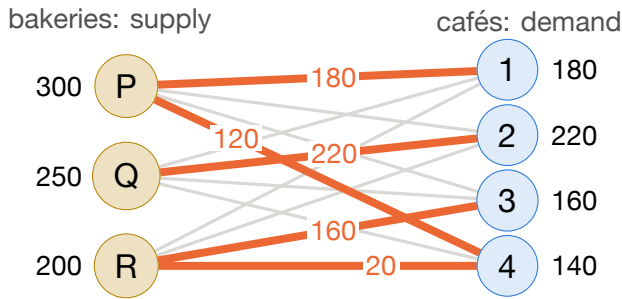
Delivery costs

baht per croissant	café 1	café 2	café 3	café 4
bakery P	4	6	9	5
bakery Q	7	3	4	8
bakery R	6	5	3	7

Each route has one decision variable.

The variable is the number of croissants sent.

The best shipping plan



The minimum cost is 2600 baht.

The numbers are croissants per day.

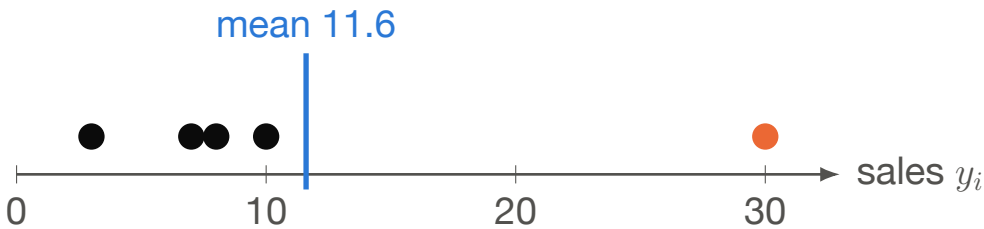
Summary: solvers

```
res = linprog(-c, A_ub=A, b_ub=b)
```

Write the problem as an LP.

A solver finds the best plan.

An outlier

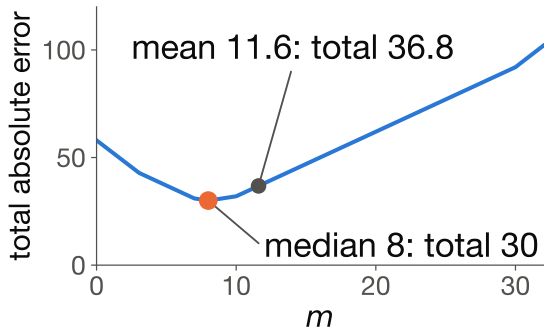


On one day, the shop sold 30 croissants.

This unusual value is an outlier.

The mean is 11.6.

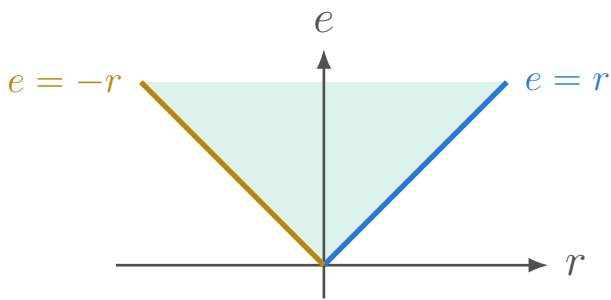
Absolute errors



Choose a number m. Add the errors $|y_i - m|$.

The total is smallest at the median, 8.

An absolute value as two constraints



$|r|$ is the smallest e with $e \geq r$ and $e \geq -r$.

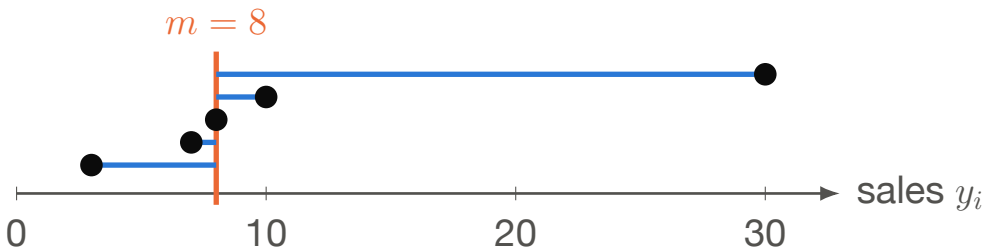
The median as an LP

Let e_i stand for the error $|y_i - m|$ of day i.

minimize $e_1 + e_2 + e_3 + e_4 + e_5$

subject to $e_i \geq y_i - m$ and $e_i \geq m - y_i$

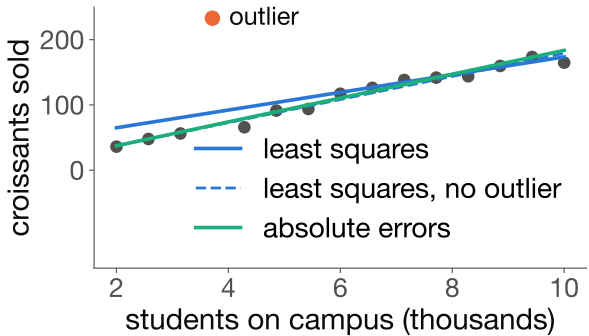
The median from linprog



linprog gives $m = 8$, the median.

The errors add up to $5 + 1 + 0 + 2 + 22 = 30$.

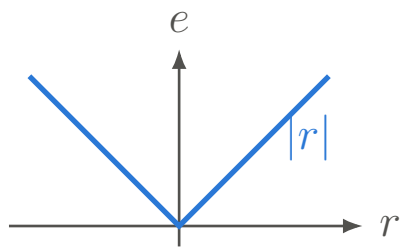
A line with absolute errors



The outlier changes the least-squares line.

The absolute-error line stays near most days.

Summary: absolute values



To minimize $|r|$, minimize e

subject to $e \geq r$ and $e \geq -r$.

Checkpoint 2

The sales on three days are 2, 5 and 9.

Which m minimizes the total absolute error?

Checkpoint 2

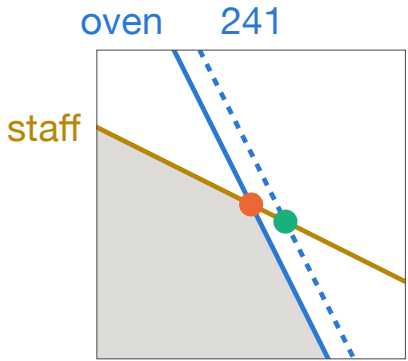
The sales on three days are 2, 5 and 9.

Which m minimizes the total absolute error?

The total is smallest at $m = 5$.

The smallest total is $3 + 0 + 4 = 7$.

One more oven minute

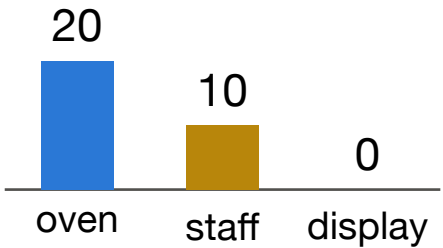


zoom near $(80, 80)$

Now the oven has 241 minutes.

The profit rises from 7200 to 7220.

Shadow price



baht per minute or place

A shadow price is the extra profit
from one more unit.
It is the most you should pay for that unit.

Unused places have shadow price 0

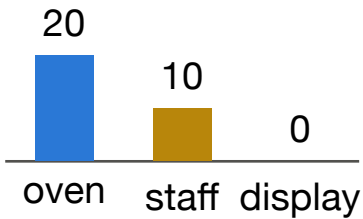


display: 100 places

The unused part, 20 places, is called slack.

A constraint with slack has shadow price 0.

Summary: shadow prices



A shadow price is the extra profit
from one more unit.

A constraint with slack has shadow price 0.

Checkpoint 3

How much profit do 10 more staff minutes add?

Checkpoint 3

How much profit do 10 more staff minutes add?

10 minutes $\times$ 10 baht per minute $= 100$ baht.

The new profit is 7300 baht.

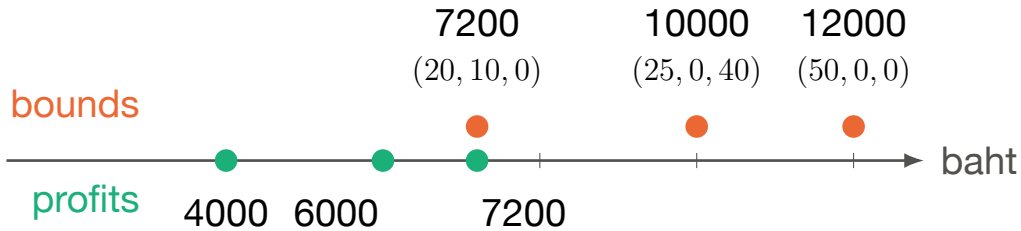
An upper bound on the profit

$$50x_1 + 40x_2 \leq 100x_1 + 50x_2 \quad \text{(because } x \geq 0\text{)}$$

$$= 50 \times (2x_1 + x_2) \leq 50 \times 240 = 12000$$

So 12000 is an upper bound on the profit.

Better bounds



Multiply each constraint by a multiplier $p_i \geq 0$.

Add the results to get an upper bound.

The smallest bound is 7200, the best profit.

The dual problem

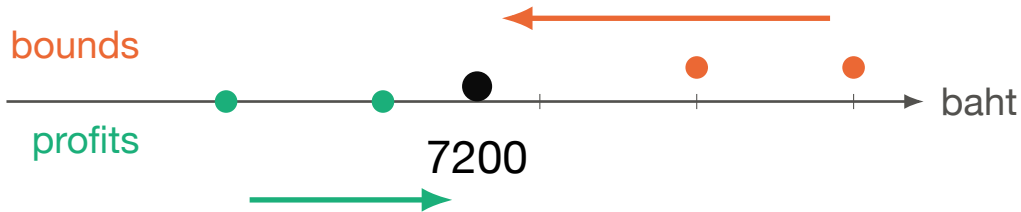
	croissant	sandwich	available	multiplier
oven	2	1	240	p_1
staff	1	2	240	p_2
display	0	1	100	p_3
profit	50	40		

The dual problem finds the smallest bound:

$$\text{minimize} \quad 240p_1 + 240p_2 + 100p_3$$

$$\text{subject to} \quad 2p_1 + p_2 \geq 50, \quad p_1 + 2p_2 + p_3 \geq 40, \quad p \geq 0$$

Duality



The best profit is also the smallest bound.

This fact is called duality.

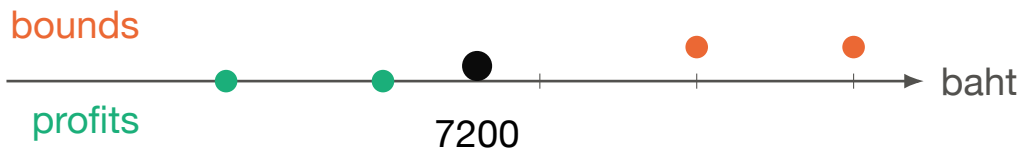
The best multipliers are the shadow prices.

Shadow prices from the solver

```
res.ineqlin.marginals
# [-20, -10, 0]
```

linprog gives the shadow prices with a minus sign.

Summary: duality



The dual problem finds the smallest bound.

Its answer gives the shadow prices.

Checkpoint 4

What upper bound does $p = (25, 10, 0)$ give?

Checkpoint 4

What upper bound does $p = (25, 10, 0)$ give?

The conditions hold, since $60 \geq 50$ and $45 \geq 40$.

The bound is $240 \times 25 + 240 \times 10 = 8400$ baht.

Where linear optimization is used



airline crews



factories



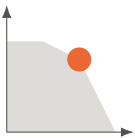
delivery



diets

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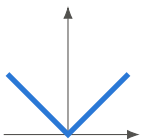
Lecture summary



LP: best plan at a vertex

```
linprog(-c,...)
```

a solver finds the plan

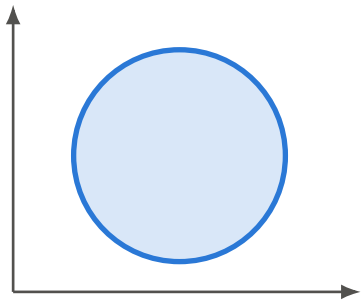


$|r|$: two linear constraints



duality: shadow prices

Next lecture



In Lecture 9, constraints can be nonlinear.

A distance constraint is one example.

Exercises

1. The oven gets 300 minutes. Find the best plan.
2. Write “minimize $|x - 3|$” as an LP.
3. What does the staff’s shadow price 10 mean?