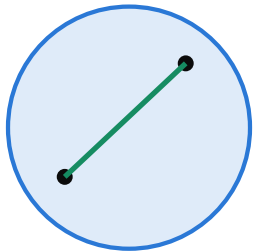
Integer optimization

208424 Lecture 10

Lecture 9

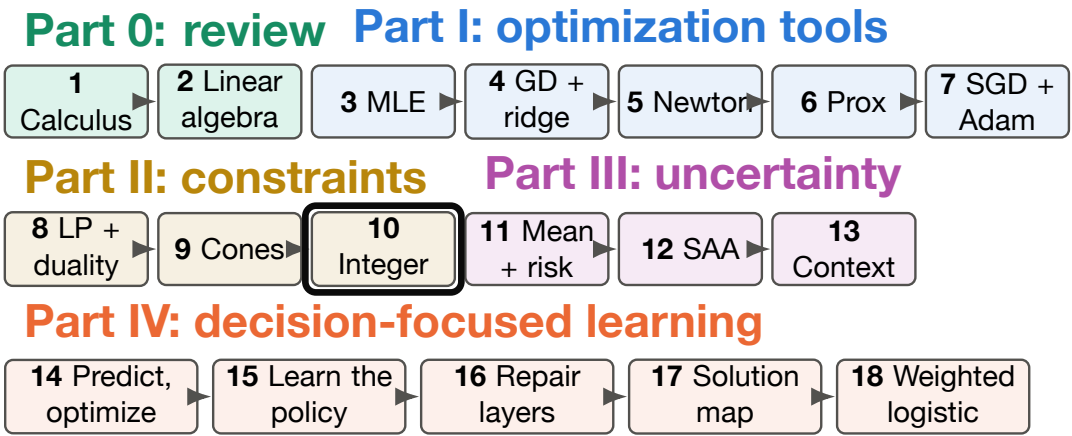


convex

In Lecture 9, every feasible region was convex.

So every local minimum was the global minimum.

Course map


Part 0: review
Part I: optimization tools
1 Calculus
2 Linear algebra
3 MLE
4 GD + ridge
5 Newton
6 Prox
7 SGD + Adam
Part II: constraints
Part III: uncertainty
8 LP + duality
9 Cones
10 Integer
11 Mean + risk
12 SAA
13 Context
Part IV: decision-focused learning
14 Predict, optimize
15 Learn the policy
16 Repair layers
17 Solution map
18 Weighted logistic

Lecture 10: Integer optimization



Photo: Scoictho, CC BY-SA 3.0, Wikimedia Commons

A bakery cannot sell half a cake.

Some decision variables must be integers.

Large and small cakes

	large cake	small cake	available
profit (baht)	500	400	
eggs	6	4	24
hours of work	1	2	6

The bakery also makes large and small cakes.

How many of each should it make each day?

Decision variables

	x_1 large cake	x_2 small cake	available
profit (baht)	500	400	
eggs	6	4	24
hours of work	1	2	6

x_1 is the number of large cakes.

x_2 is the number of small cakes.

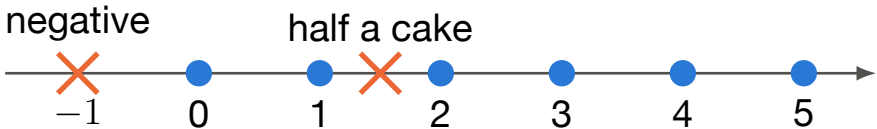
The cake problem

maximize $500x_1 + 400x_2$

subject to $6x_1 + 4x_2 \leq 24$ (eggs)

$x_1 + 2x_2 \leq 6$ (hours)

Integer constraints



As in Lecture 8, $x_1 \geq 0$ and $x_2 \geq 0$.

Now x_1 and x_2 must also be integers.

These are called integer constraints.

A mixed-integer linear problem (MILP)

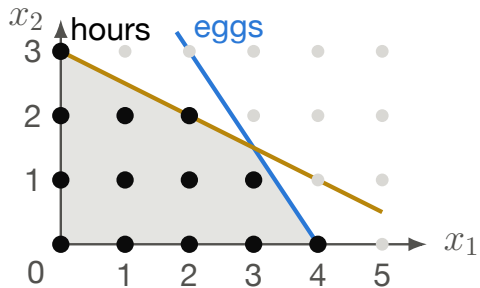


An LP with integer constraints is an MILP.

Some or all of its variables must be integers.

The cake problem is an MILP.

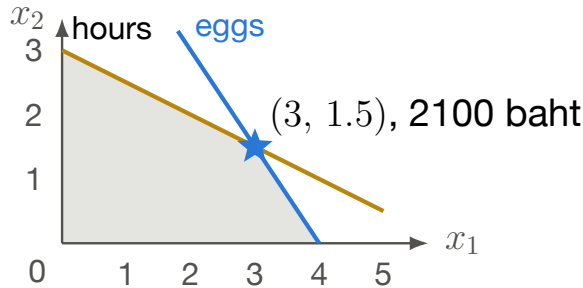
The feasible integer plans



Each black dot is a feasible integer plan.

There are 13 feasible integer plans.

The LP relaxation

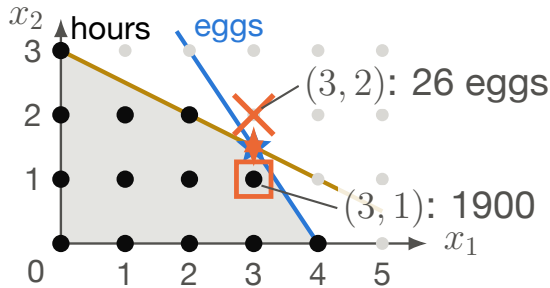


Without the integer constraints, we get an LP.

This LP is called the LP relaxation.

Its optimal solution is $(3, 1.5)$.

Rounding the LP solution

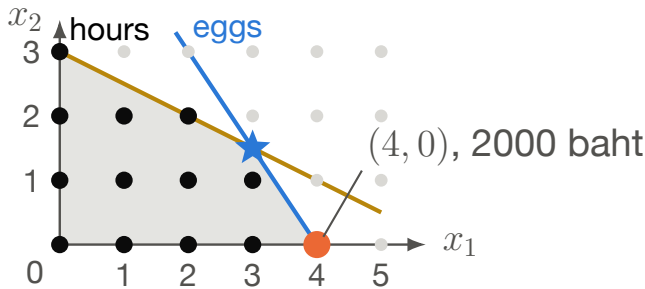


Rounding down gives $(3, 1)$, with profit 1900.

Rounding up gives $(3, 2)$.

It needs 26 eggs, so it is infeasible.

The optimal integer plan

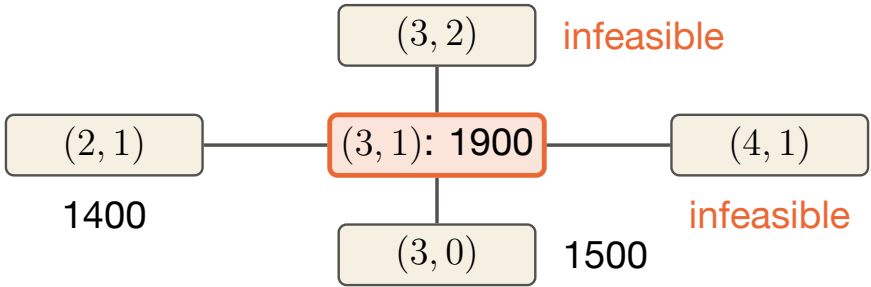


The optimal integer plan is $(4, 0)$.

We also call it the global maximum.

Rounding $(3, 1.5)$ never gives $(4, 0)$.

A local maximum

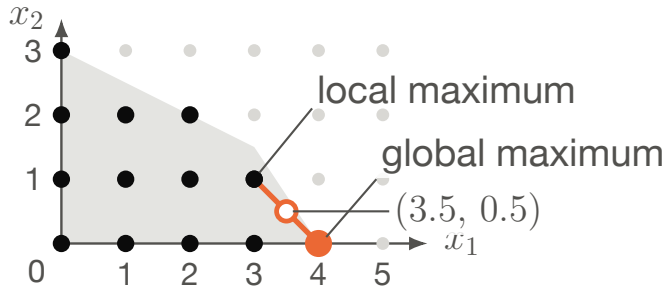


A neighbor of $(3, 1)$ has x_1 or x_2 changed by 1.

Every feasible neighbor has less profit.

So $(3, 1)$ is called a local maximum.

Integer plans are not convex

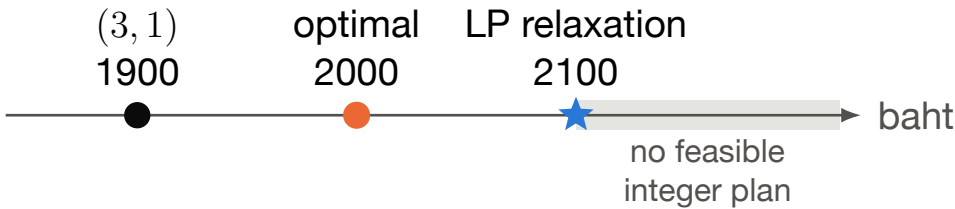


The midpoint $(3.5, 0.5)$ is not an integer plan.

So the set of feasible integer plans is not convex.

A local maximum may not be the global maximum.

An upper bound from the LP relaxation

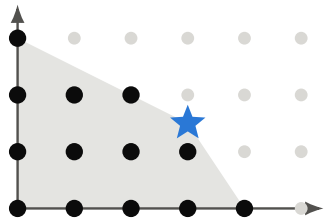


The LP relaxation allows every feasible integer plan.

No feasible integer plan has a profit above 2100.

So 2100 is an upper bound on the profit.

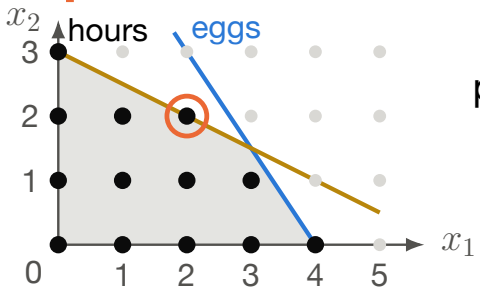
Summary: integer constraints



Rounding the LP solution may not give the optimal integer plan.

The LP relaxation gives an upper bound.

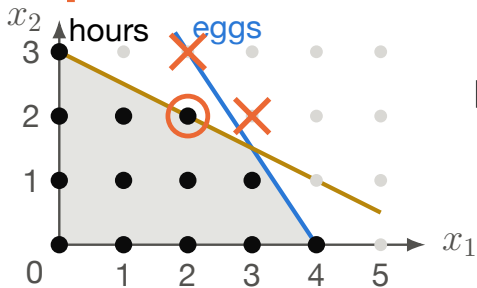
Checkpoint 1



profit $= 500x_1 + 400x_2$

Is $(2, 2)$ a local maximum?

Checkpoint 1



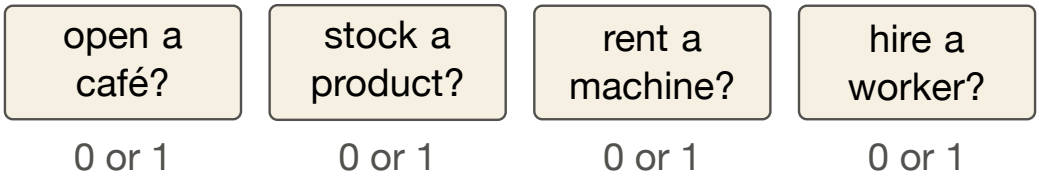
profit $= 500x_1 + 400x_2$

Is $(2, 2)$ a local maximum?

Yes. Every feasible neighbor gives less than 1800.

$(1, 2)$ gives 1300, and $(2, 1)$ gives 1400.

Yes-or-no decisions



Many decisions have the answer yes or no.

A binary variable can only be 0 or 1.

$y = 1$ means yes, and $y = 0$ means no.

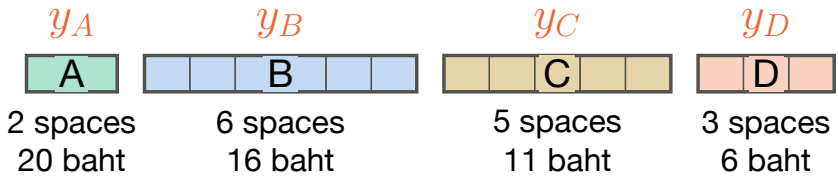
A café shelf



A café shelf has 11 spaces.

Which products should the café stock?

Decision variables for the shelf



$y_A = 1$ if the café stocks A, and 0 if not.

y_B, y_C and y_D are defined in the same way.

$y_A + y_B + y_C + y_D$ is the number of products stocked.

The shelf problem

maximize $\underbrace{20y_A + 16y_B + 11y_C + 6y_D}_{\text{profit (baht a day)}}$

subject to $\underbrace{2y_A + 6y_B + 5y_C + 3y_D}_{\text{spaces}} \leq 11$

Each y is 0 or 1.

A knapsack problem



a knapsack



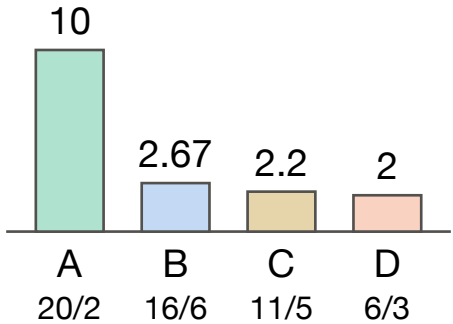
the shelf

Photo: Pegasovagante, CC0, Wikimedia Commons.

The shelf problem is a knapsack problem.

The name comes from packing a bag.

The shelf's LP relaxation

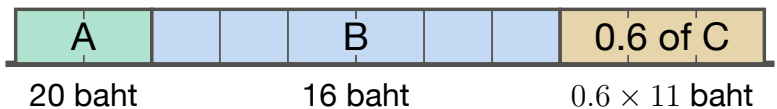


profit per space

The LP relaxation allows any y from 0 to 1.

The LP fills the shelf in order of profit per space.

The LP solution for the shelf

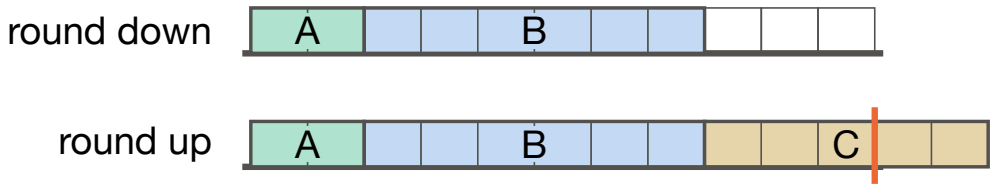


A and B use 8 spaces, and C uses the last 3.

So $y_C = 3/5 = 0.6$.

The upper bound is $20 + 16 + 0.6 \times 11 = 42.6$.

Rounding the shelf's LP solution

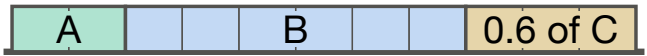


Rounding down gives A and B, with profit 36.

Rounding up gives A, B and C.

They need 13 spaces, but the shelf has 11.

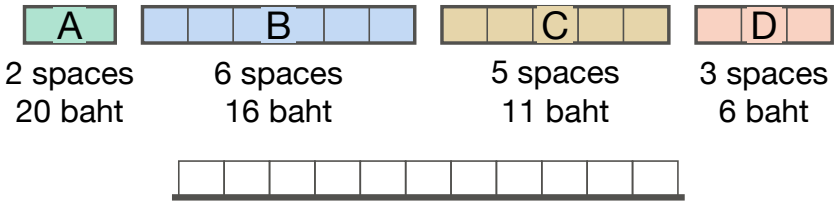
Summary: binary decisions



A binary variable is 0 (no) or 1 (yes).

The shelf problem is a knapsack problem.

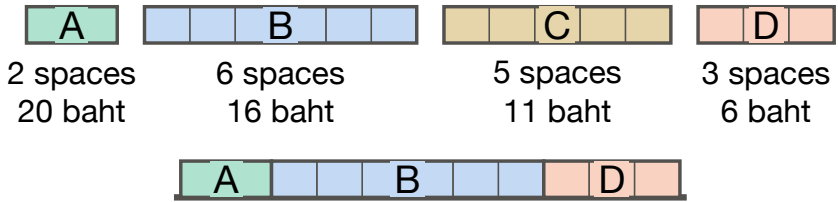
Checkpoint 2



The upper bound is 42.6.

Which choice of products is optimal, and why?

Checkpoint 2



The upper bound is 42.6.

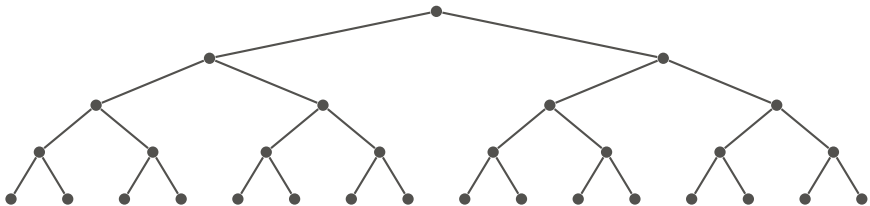
Which choice of products is optimal, and why?

A, B and D give 42, and profits are integers.

No profit is above 42.6, so none is above 42.

Too many choices to check

A
B
C
D

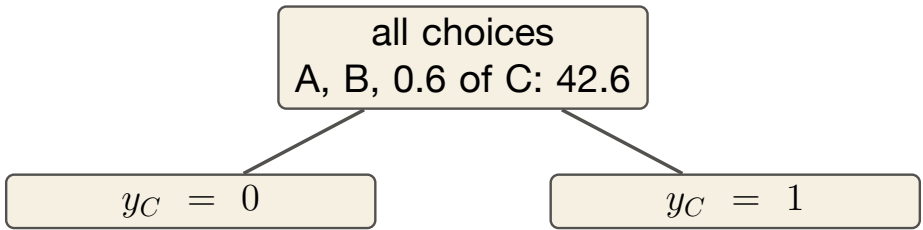


4 products give $2 \times 2 \times 2 \times 2 = 16$ choices.

50 products give 2^{50}, about 10^{15} choices.

Checking every choice takes too long.

Branching

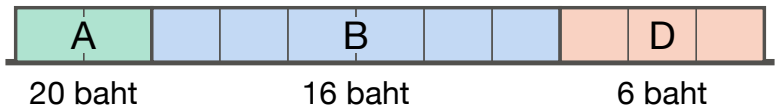


In the LP solution, $y_C = 0.6$ is a fraction.

Branching splits the choices into $y_C = 0$ and $y_C = 1$.

Each part is called a subproblem.

The subproblem $y_C = 0$

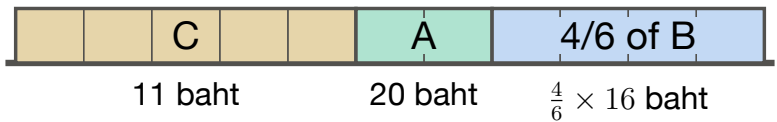


Without C, the LP solution is A, B and D.

Every y is 0 or 1, and the profit is 42.

This is the best choice so far.

The subproblem $y_C = 1$

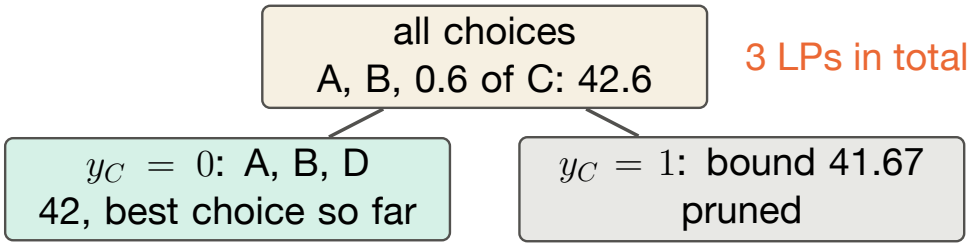


With C, the LP solution is C, A and 4/6 of B.

Its upper bound is $11 + 20 + \frac{4}{6} \times 16 = 41.67$.

$41.67 < 42$, so we prune (drop) this subproblem.

Branch and bound on the shelf



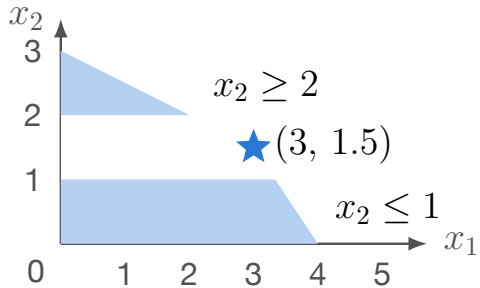
3 LPs in total

Every choice has $y_C = 0$ or $y_C = 1$.

So A, B and D, with profit 42, is optimal.

This method is called branch and bound.

Branching in the cake problem

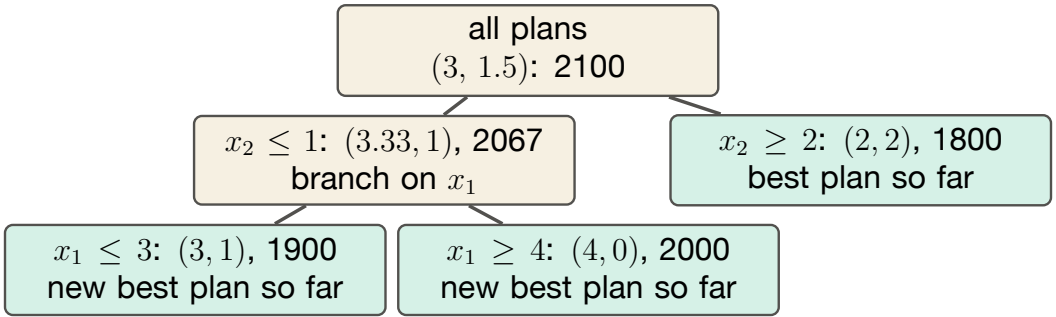


In the cake problem, $x_2 = 1.5$ is a fraction.

No integer lies between 1 and 2.

So we split into $x_2 \leq 1$ and $x_2 \geq 2$.

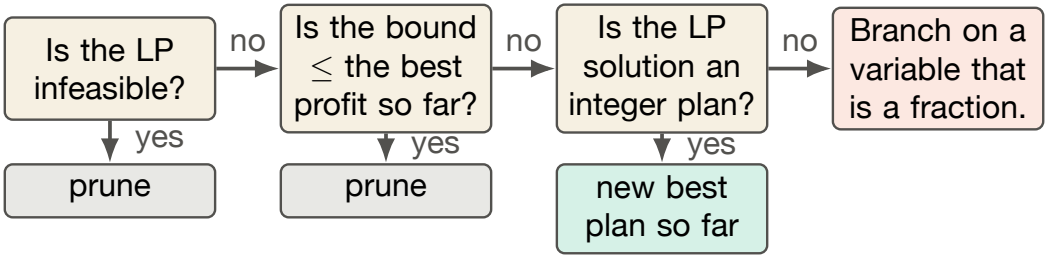
Branch and bound on the cakes



$(4, 0)$, with profit 2000, is optimal.

We solved 5 LPs.

The steps of branch and bound



For each subproblem, we first solve its LP.

We repeat until no subproblem is left.

Then the best plan so far is optimal.

Solving the shelf with milp

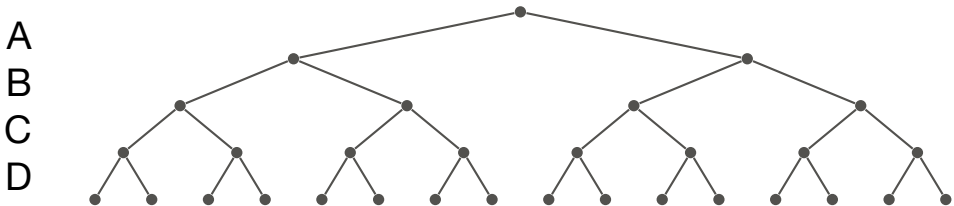
```
from scipy.optimize import milp, Bounds
from scipy.optimize import LinearConstraint
space = LinearConstraint([[2, 6, 5, 3]], ub=11)
res = milp(c=[-20, -16, -11, -6],
   integrality=[1, 1, 1, 1],
   bounds=Bounds(0, 1), constraints=space)
res.x      # [1, 1, 0, 1]: A, B, D
-res.fun   # 42
```

milp minimizes, so we give it $-$profit.

Each 1 in `integrality` makes that y an integer.

`Bounds(0, 1)` means that $0 \leq y \leq 1$.

How milp solves an MILP

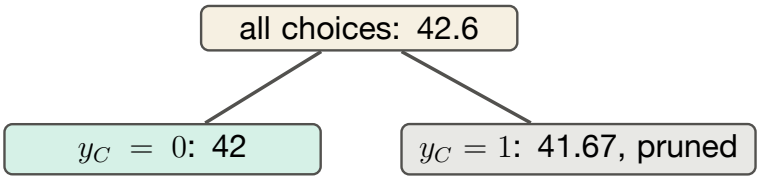


milp calls the solver HiGHS.

HiGHS uses a form of branch and bound.

No known method solves every MILP fast.

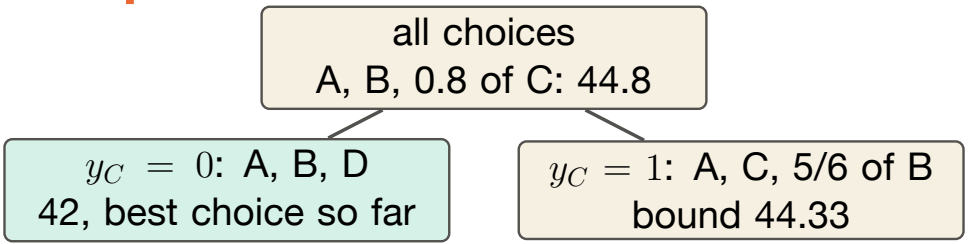
Summary: branch and bound



We branch on a variable that is a fraction.

We prune a subproblem if its upper bound is not above the best profit so far.

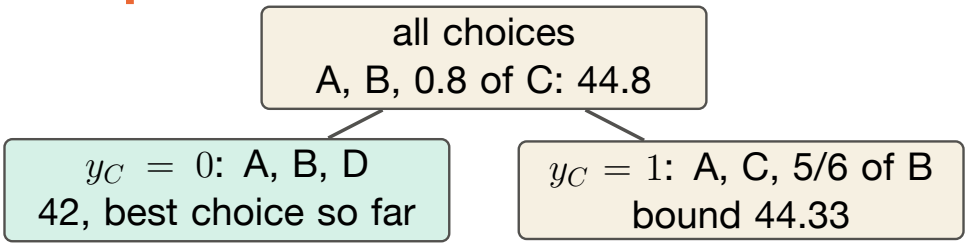
Checkpoint 3



The shelf now has 12 spaces.

Should we prune the subproblem $y_C = 1$?

Checkpoint 3



The shelf now has 12 spaces.

Should we prune the subproblem $y_C = 1$?

No. Its bound 44.33 is above 42.

We branch on y_B, which is 5/6.

Pies





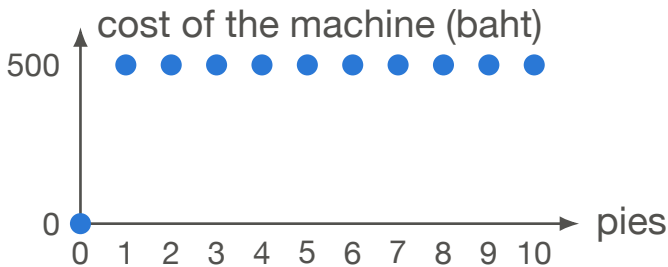


	croissant	sandwich	pie	available
profit (baht)	50	40	90	
oven minutes	2	1	3	240
staff minutes	1	2	2	240
display spaces	0	1	0	100

Photos (Wikimedia Commons): Petr Kratochvil, CC0; Andy Li, CC0; Sarah Stierch (Missvain), CC0.

The bakery of Lecture 8 can also make pies.

A fixed cost



Pies need a machine, which costs 500 baht a day.

The cost is the same for 1 pie or 10 pies.

This is called a fixed cost.

Decision variables with pies



x_1 croissants



x_2 sandwiches



z pies



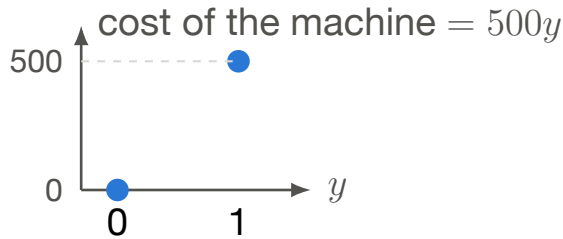
y machine

z is the number of pies made each day.

$y = 1$ if we rent the machine, and $y = 0$ if not.

x_1, x_2 and z must be integers ≥ 0.

The profit with pies



maximize $50x_1 + 40x_2 + 90z - 500y$

We pay 500 baht only when $y = 1$.

The constraints with pies

$$2x_1 + x_2 + 3z \leq 240 \quad \text{(oven)}$$

$$x_1 + 2x_2 + 2z \leq 240 \quad \text{(staff)}$$

$$x_2 \leq 100 \quad \text{(display)}$$

Pies without the machine

$$x_1 = 0,\ \ x_2 = 60,\ \ z = 60,\ \ y = 0$$

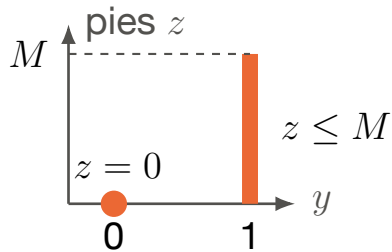
profit 7800 baht

Can the solver make pies with $y = 0$?

Yes. No constraint connects z and y yet.

The plan has 60 pies but no machine.

The big-M constraint

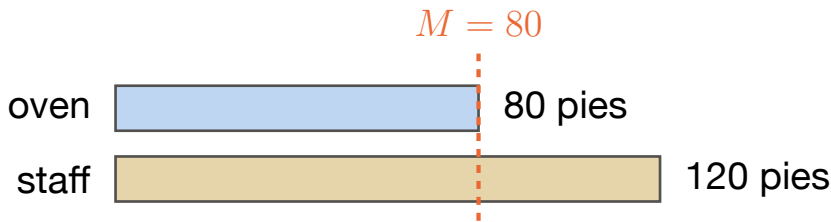


$$z \leq My$$

In this big-M constraint, M is a large constant.

If $y = 0$, then $z \leq 0$, so we make no pies.

How large must M be?



Oven time allows at most $240/3 = 80$ pies.

Staff time allows at most $240/2 = 120$ pies.

No plan has more than 80 pies, so we take $M = 80$.

If M is too small

M	pies	machine	profit (baht)
20	0	no	7200
80	60	yes	7300

With $M = 20$, a plan with 60 pies is infeasible.

So the solver returns a worse plan.

The best plan with pies



0



60



60



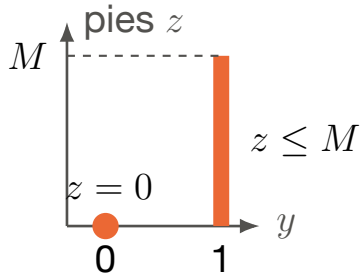
rented

The best plan makes 60 sandwiches and 60 pies.

The profit is 7300 baht.

Without pies, the best profit was 7200 baht.

Summary: fixed costs



A fixed cost needs a binary variable y.

$z \leq My$ allows pies only when $y = 1$.

Checkpoint 4



$y = 1$ if we rent the truck.

w is the number of croissants it sells.

Photo: Kurtkaiser, CC0, Wikimedia Commons.

A food truck sells at most 90 croissants a day.

Write the big-M constraint.

Checkpoint 4



$y = 1$ if we rent the truck.

w is the number of croissants it sells.

Photo: Kurtkaiser, CC0, Wikimedia Commons.

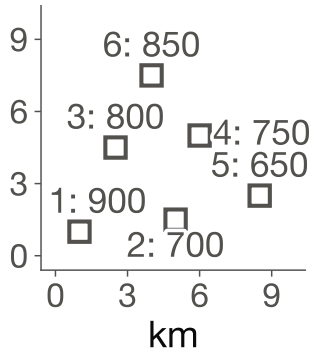
A food truck sells at most 90 croissants a day.

Write the big-M constraint.

$$w \leq 90y$$

If $y = 0$, the truck sells no croissants.

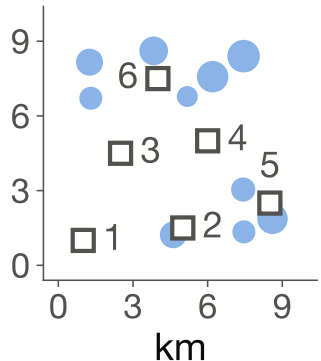
Six possible sites



A bakery can open cafés at 6 sites.

Each site has a daily opening cost in baht.

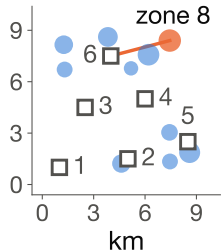
Ten zones of customers



The customers are in 10 zones.

A larger circle means more demand.

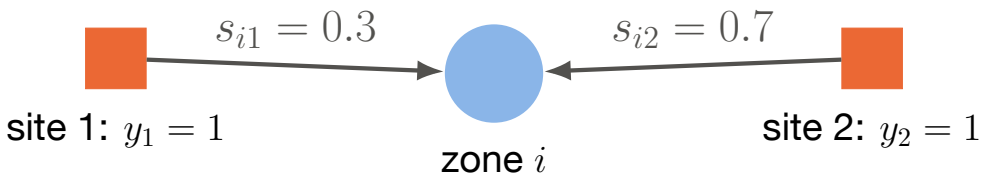
Delivery costs



$$c_{8,6} = \underbrace{2}_{\substack{\text{baht per} \\ \text{croissant per km}}} \times \underbrace{3.56}_{\text{km}} \times \underbrace{58}_{\substack{\text{croissants} \\ \text{a day}}} = \underbrace{413}_{\substack{\text{baht} \\ \text{a day}}}$$

c_{ij} is the cost if site j serves all of zone i.

Decision variables for the cafés



$y_j = 1$ if we open site j, and $y_j = 0$ if not.

s_{ij} is the share of zone i's demand served by site j.

s_{ij} can be any number from 0 to 1.

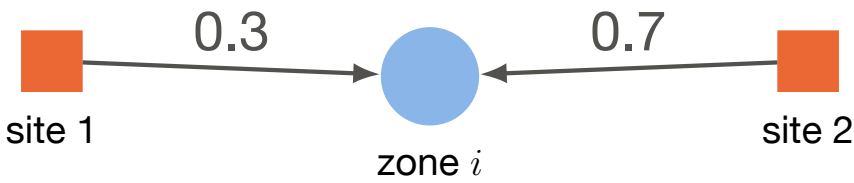
A facility location problem

$$\text{minimize} \quad \underbrace{\sum_{j=1}^{6} f_j y_j}_{\text{opening costs}} + \underbrace{\sum_{i=1}^{10} \sum_{j=1}^{6} c_{ij} s_{ij}}_{\text{delivery costs}}$$

f_j is the opening cost of site j.

This is called a facility location problem.

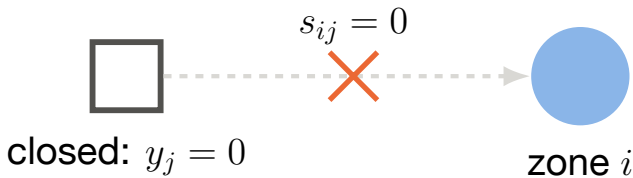
Each zone gets all of its demand



$$s_{i1} + s_{i2} + \cdots + s_{i6} = 1$$

The shares of zone i add up to 1.

Only open sites serve zones

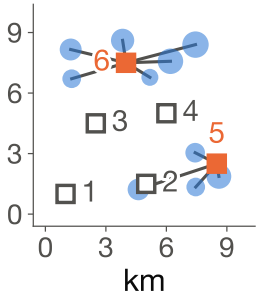


$$s_{ij} \leq y_j$$

Site j serves zone i only if site j is open.

This is a big-M constraint with $M = 1$.

The optimal cafés

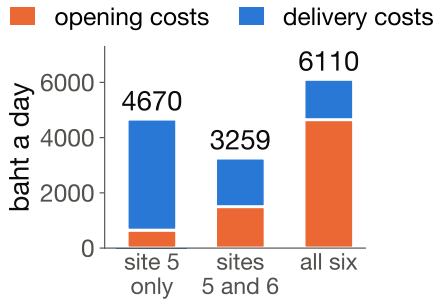


The optimal plan opens sites 5 and 6.

The total cost is 3259 baht a day.

Each zone is served by its nearest open site.

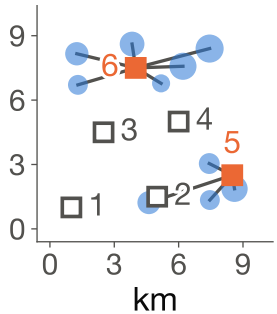
Opening costs and delivery costs



With more open sites, opening costs rise and delivery costs fall.

No other choice of sites costs less than 3259.

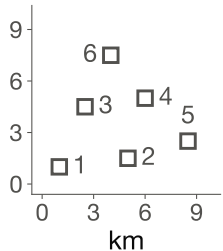
Summary: facility location



In facility location, we choose sites to open.

With $s_{ij} \leq y_j$, only open sites serve zones.

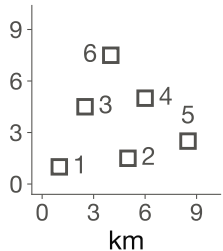
Checkpoint 5



The bakery can open at most 3 sites.

Write this as a constraint.

Checkpoint 5



The bakery can open at most 3 sites.

Write this as a constraint.

$$y_1 + y_2 + \cdots + y_6 \leq 3$$

The sum is the number of open sites.

Where integer optimization is used



staff schedules



train timetables



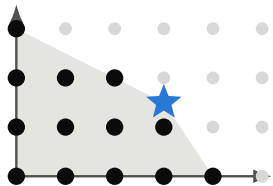
power plants

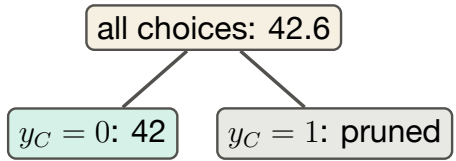


store locations

Photos (Wikimedia Commons, cropped): Beendy234, CC0; Inklein, CC0; Gretar Ívarsson, public domain; Nyapotheosis, CC0.

Lecture summary



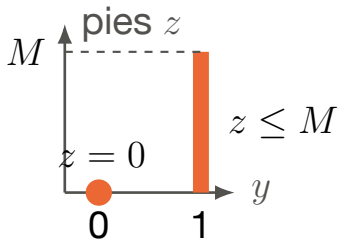


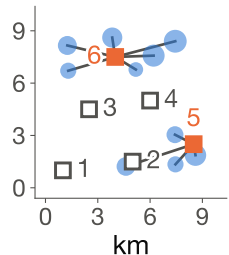
The feasible integer plans are not convex.

The LP relaxation gives an upper bound.

Branch and bound uses these bounds to prune.

Lecture summary (continued)

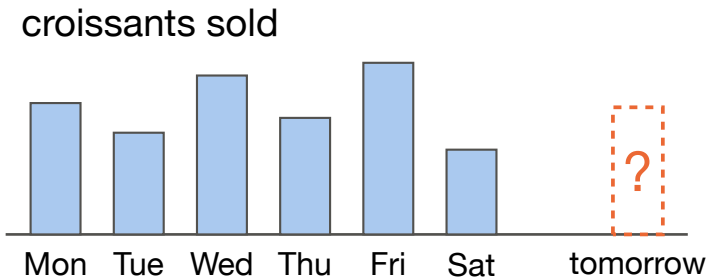




A fixed cost needs the big-M constraint $z \leq My$.

Facility location is an MILP.

Next lecture



So far, every number in a problem was known.

In Lecture 11, the demand is random.

Exercises

1. Write “open site 2 only if site 1 is open”.
2. Solve the shelf with 12 spaces using milp.
3. With 150 staff minutes, how large must M be?