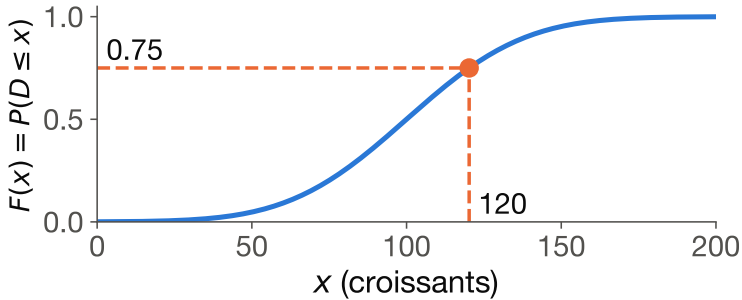
Sample average approximation

208424 Lecture 12

Lecture 11: the optimal order

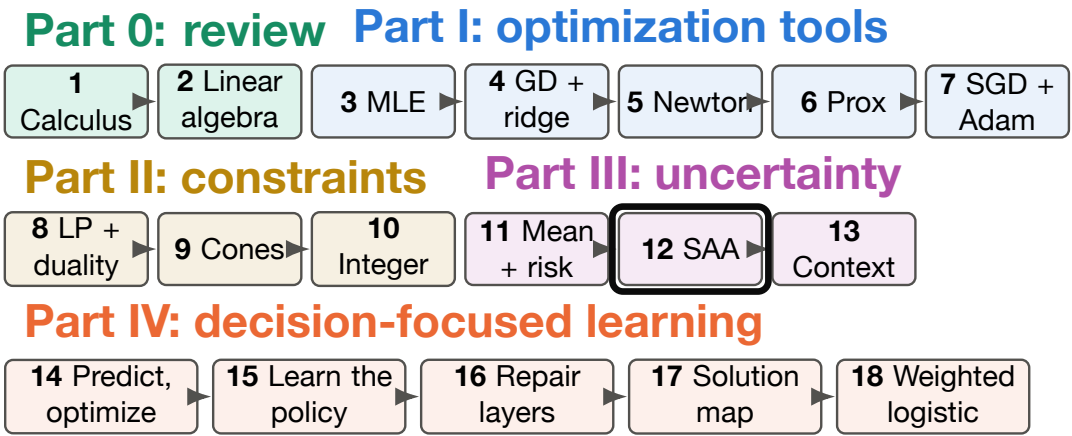


In Lecture 11, we knew the distribution of D.

The optimal order was the 0.75 quantile of D.

Today, we know D only on past days.

Course map


Part 0: review
Part I: optimization tools
1 Calculus
2 Linear algebra
3 MLE
4 GD + ridge
5 Newton
6 Prox
7 SGD + Adam
Part II: constraints
Part III: uncertainty
8 LP + duality
9 Cones
10 Integer
11 Mean + risk
12 SAA
13 Context
Part IV: decision-focused learning
14 Predict, optimize
15 Learn the policy
16 Repair layers
17 Solution map
18 Weighted logistic

Lecture 12: Sample average approximation

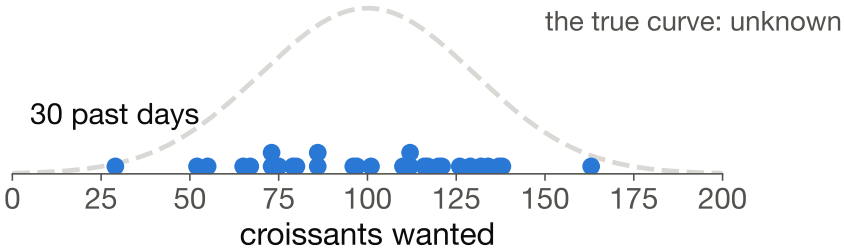


Photo: nappa, CC BY 2.0, Wikimedia Commons

A café knows the demand only on past days.

An investor knows the returns only on past months.

Past days of demand

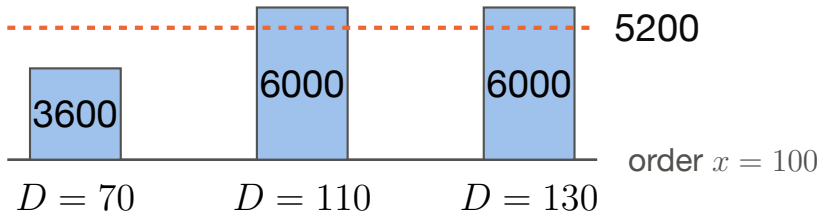


The café recorded the demand on 30 past days.

These 30 days are a sample of the demand.

We choose the order from this sample.

One order on three past days



On three past days, D was 70, 110 and 130.

Ordering 100 gives 3600, 6000 and 6000 baht.

The average profit is 5200 baht.

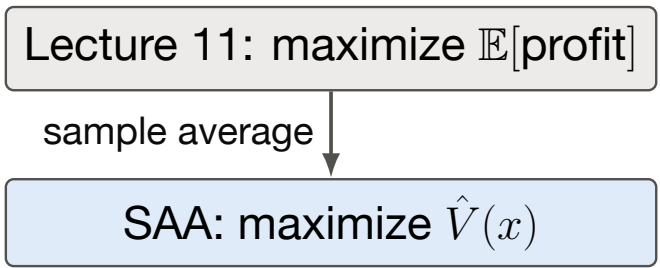
The average profit on past days

We write $\hat{V}(x)$ for the average profit on past days.

$$\hat{V}(x) = \underbrace{\frac{1}{n}\sum_{i=1}^{n}}_{\text{average over the } n \text{ past days}} \underbrace{\big(80\min(D_i, x) - 20x\big)}_{\text{profit on day } i}$$

The demand on past day i is D_i.

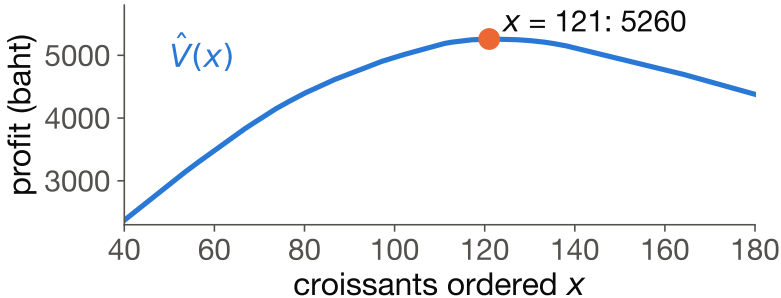
Sample average approximation (SAA)



SAA maximizes $\hat{V}(x)$ instead of $\mathbb{E}$[profit].

The sample average $\hat{V}(x)$ approximates $\mathbb{E}$[profit].

$\hat{V}$ for the 30 past days

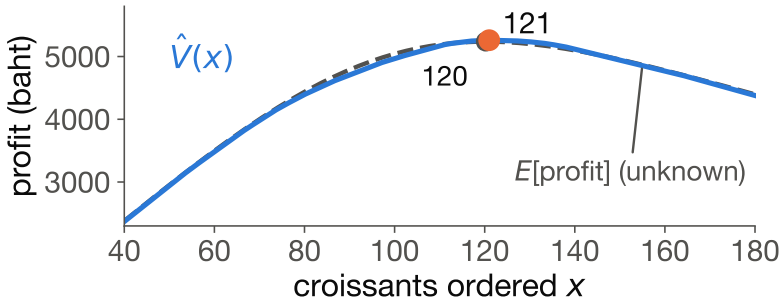


Each point is the average profit of one order.

The maximum of $\hat{V}$ is 5260 baht, at $x = 121$.

So the SAA order is 121 croissants.

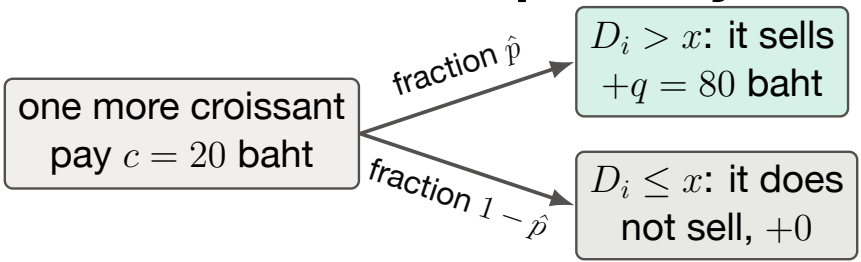
$\hat{V}$ and the expected profit



The dashed curve is $\mathbb{E}[\text{profit}]$ from Lecture 11.

Its maximum is at 120, close to the SAA order 121.

One more croissant on past days

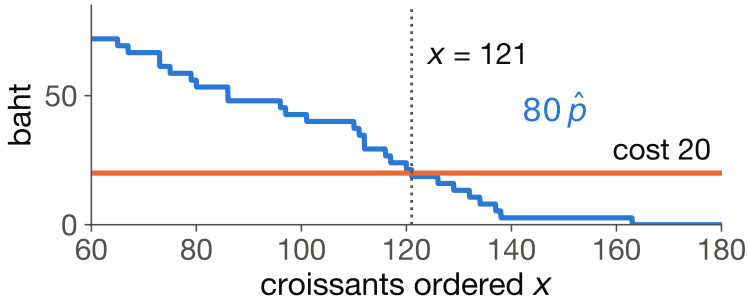


One more croissant costs 20 baht.

That croissant sells only on days with $D_i > x$.

average extra profit $= 80\,\underbrace{\hat{p}}_{\text{fraction of past days with } D_i > x} - 20$

Where to stop on past days

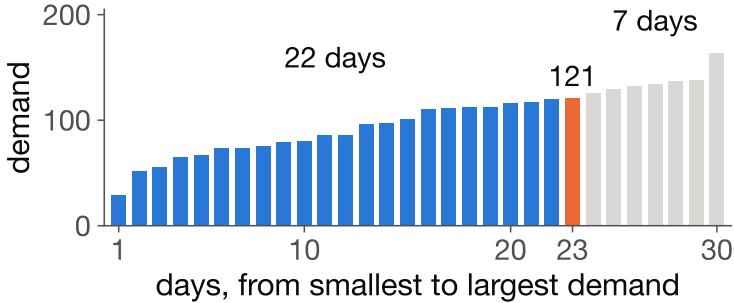


We add croissants while $80\,\hat{p} > 20$, that is, $\hat{p} > 0.25$.

At $x = 121$, only 7 of the 30 days have $D_i > x$.

Since $7/30 < 0.25$, we stop at 121.

Finding the SAA order by sorting

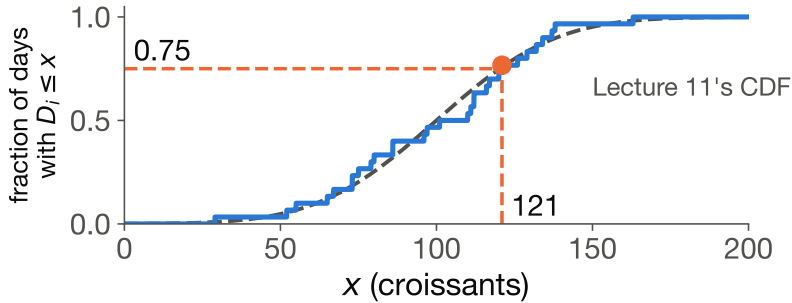


Sort the 30 past demands from smallest to largest.

Since $0.75 \times 30 = 22.5$, we take the 23rd value.

The 23rd value is 121, so the SAA order is 121.

The sample 0.75 quantile

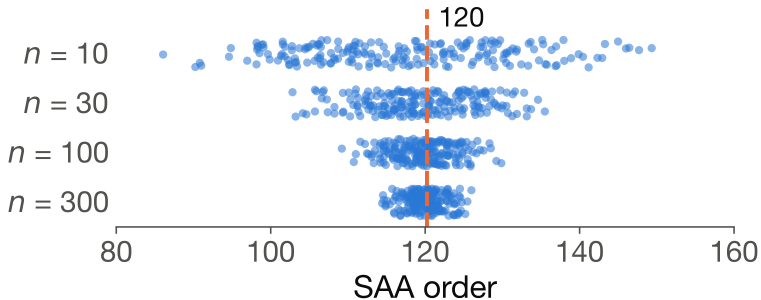


The steps show the fraction of days with $D_i \leq x$.

This fraction first goes above 0.75 at $x = 121$.

So 121 is called the sample 0.75 quantile.

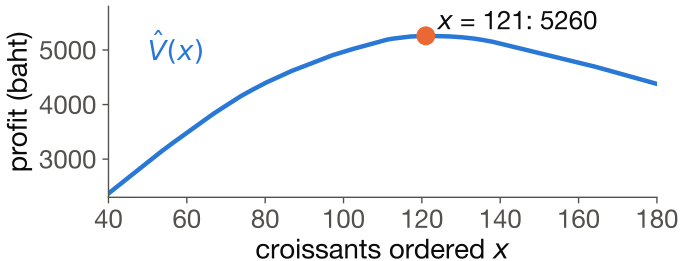
More past days give a better order



Each dot is the SAA order of one simulated café.

With more past days, the orders are closer to 120.

Summary: SAA



SAA maximizes the average profit on past days.

The SAA order is the sample $(q - c)/q$ quantile.

Checkpoint 1

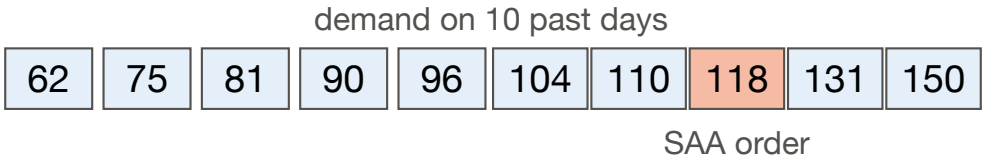
demand on 10 past days

62	75	81	90	96	104	110	118	131	150

The cost is 20 baht, and the price is 80 baht.

What is the SAA order for these 10 past days?

Checkpoint 1



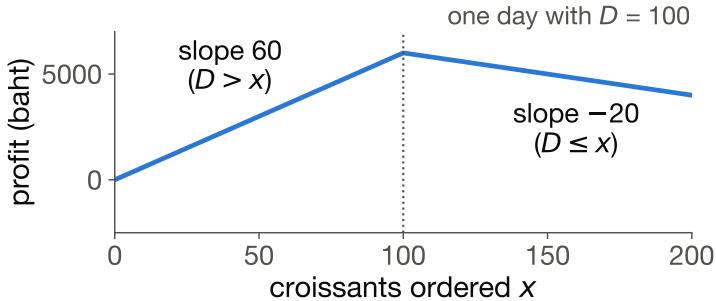
The cost is 20 baht, and the price is 80 baht.

What is the SAA order for these 10 past days?

Since $0.75 \times 10 = 7.5$, we take the 8th value.

So the SAA order is 118 croissants.

The slope of one day's profit

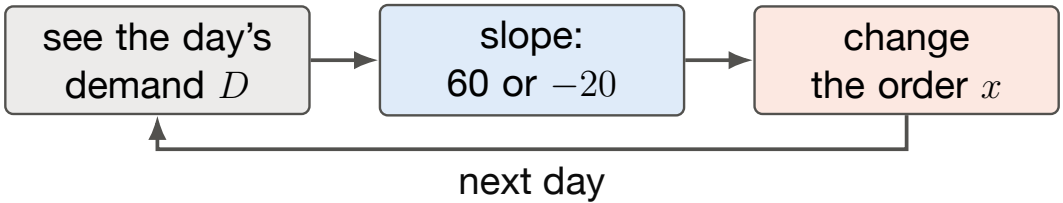


If $D > x$, one more croissant adds 60 baht.

If $D \leq x$, one more croissant loses 20 baht.

So the slope of one day's profit is 60 or -20.

SGD for the order (Lecture 7)

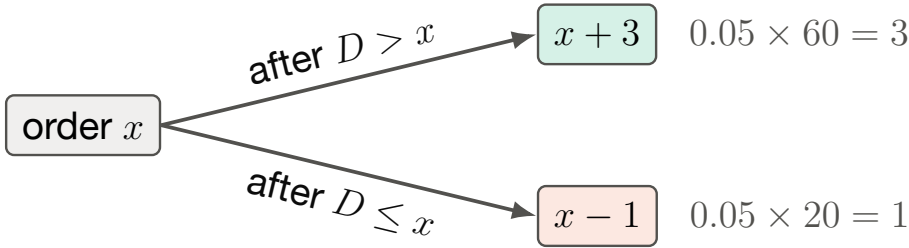


SGD changes the order after each day.

$$x \leftarrow x + \eta \times (\text{slope of the day's profit})$$

We add the slope because we maximize profit.

SGD steps with $\eta = 0.05$

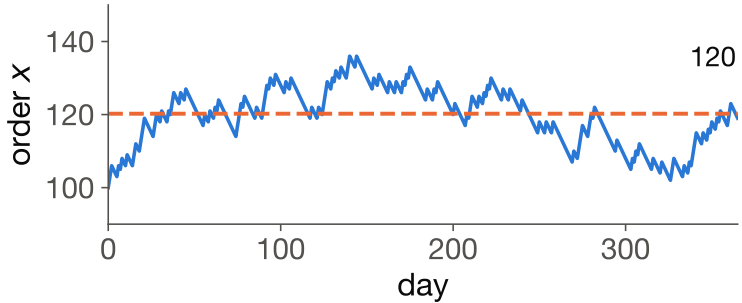


We use the step size $\eta = 0.05$.

After a day with $D > x$, the order goes up by 3.

After a day with $D \leq x$, the order goes down by 1.

SGD over one year

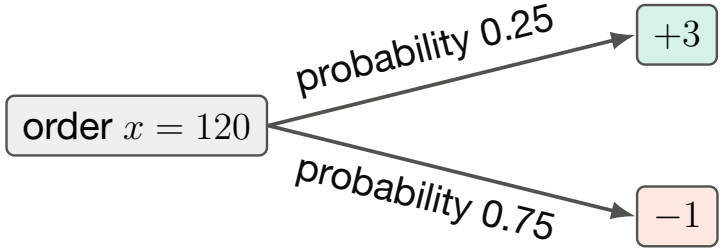


The café starts with an order of 100.

After about 20 days, the order is near 120.

Then the order moves up and down around 120.

Why the order stays near 120

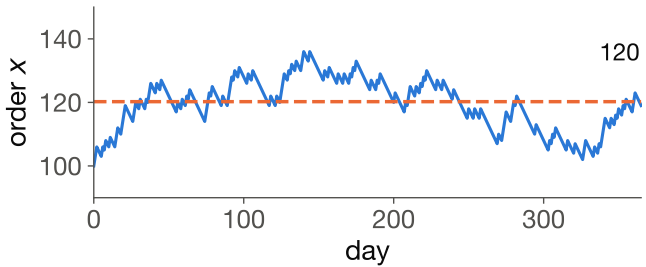


At $x = 120$, $P(D > x) = 0.25$ (Lecture 11).

average change $= 0.25 \times 3 - 0.75 \times 1 = 0$

So the order stays near 120.

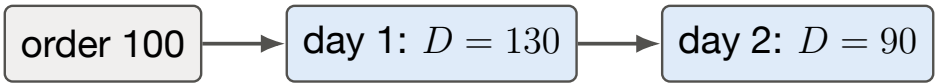
Summary: SGD



SGD changes the order a little after each day.

SGD does not need to store the past days.

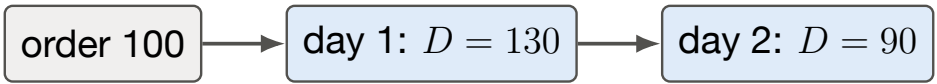
Checkpoint 2



The order is 100, and the step size is 0.05.

What is the order after day 1 and after day 2?

Checkpoint 2



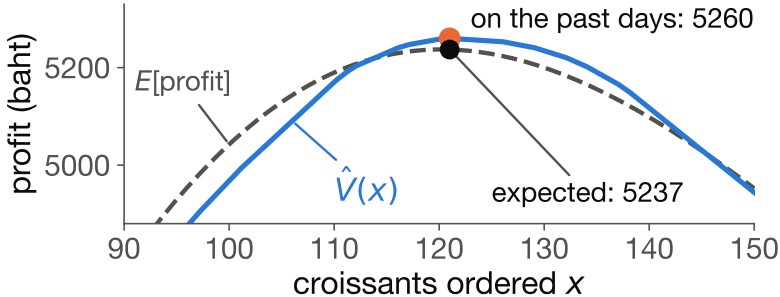
The order is 100, and the step size is 0.05.

What is the order after day 1 and after day 2?

After day 1, the order is $100 + 3 = 103$.

After day 2, the order is $103 - 1 = 102$.

Profit on the past days

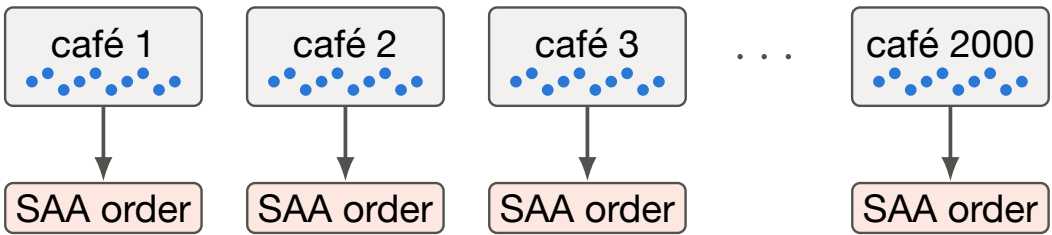


On its 30 past days, the SAA order earns 5260 baht.

Its expected profit is 5237 baht.

So the profit on the past days is too high.

Many simulated cafés



The computer simulates 2000 cafés.

Each café has 10 past days and its own SAA order.

Profit on the past days is too high

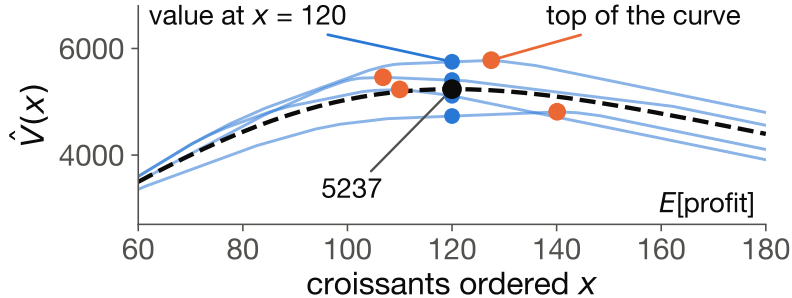


On past days, SAA orders earn 5310 on average.

No order has an expected profit above 5237 baht.

The expected profit of SAA orders is 5173 baht.

Why the past days look too good

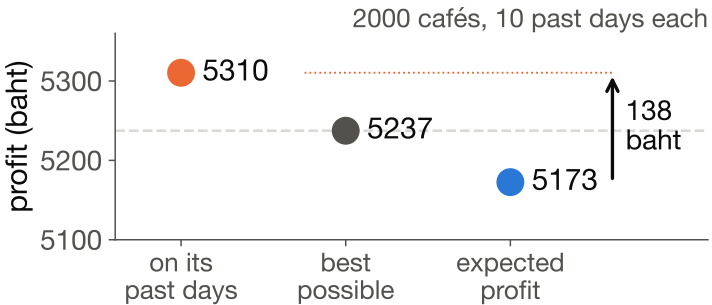


On average, $\hat{V}(120)$ is 5237 baht.

Each curve's top is at least its value at 120.

So on average, the tops are above 5237 baht.

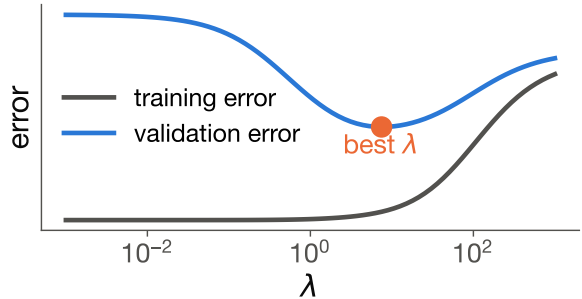
How much too high?



Profit on the past days is too high on average.

This profit is 138 baht above the expected profit.

Training error is too low (Lecture 4)

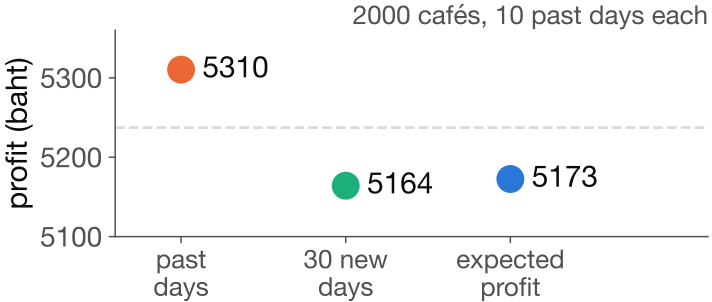


In Lecture 4, training error was too low.

The model was fitted on the training data.

The SAA order was also chosen on the past days.

Evaluate on new days

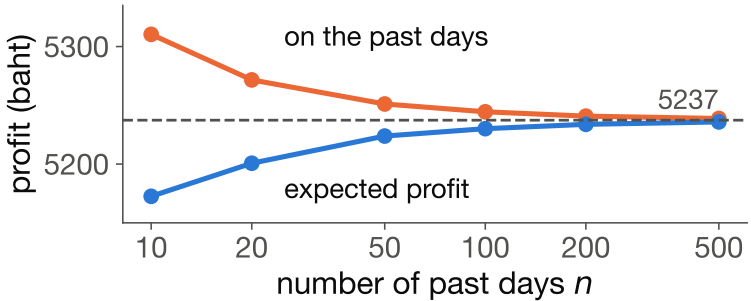


On 30 new days, the average profit is 5164 baht.

Profit on new days is unbiased (Lecture 7).

So we evaluate an order on new days.

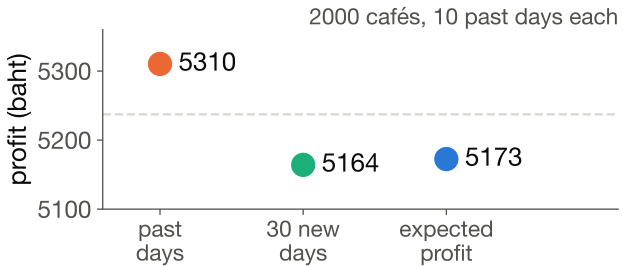
More past days, a smaller difference



With more past days, both curves get close to 5237.

With 100 days, the curves differ by only 14 baht.

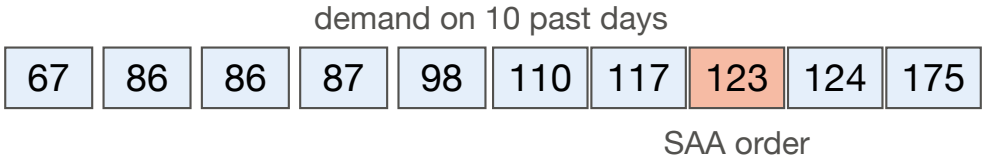
Summary: past days and new days



Profit on the past days is too high on average.

We evaluate an order on new days.

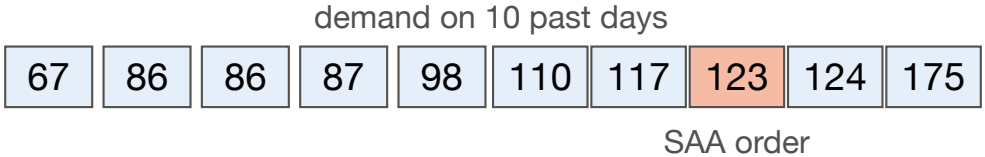
Checkpoint 3



On 10 past days, the SAA order earns 5700 baht.

Should the café expect this profit on new days?

Checkpoint 3



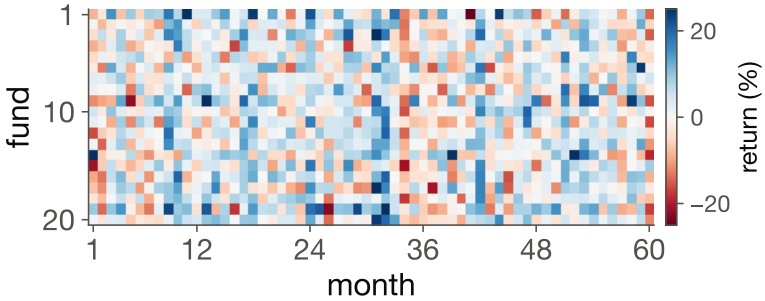
On 10 past days, the SAA order earns 5700 baht.

Should the café expect this profit on new days?

No. No order has an expected profit above 5237.

The expected profit of this order is 5234 baht.

20 funds, 60 months

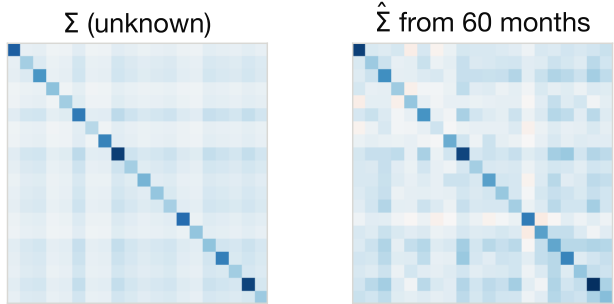


We have 60 months of returns for 20 funds.

Each return is for one month, not one year.

We do not know the true covariance matrix Σ.

The sample covariance matrix



SAA estimates Σ by averages over the 60 months.

The result is the sample covariance matrix $\hat{\Sigma}$.

We estimate 20 variances and 190 covariances.

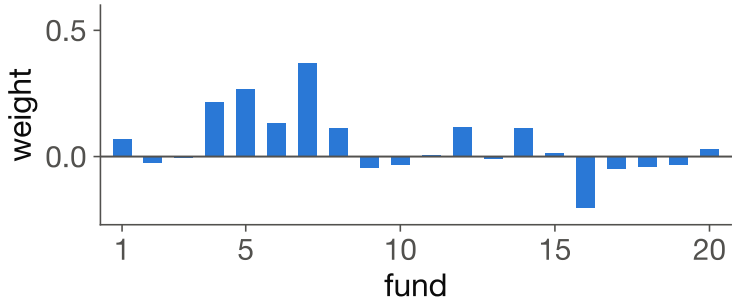
Minimum variance from past months

$$\text{minimize } \underbrace{v^\top}_{\text{the 20 weights}} \hat{\Sigma} v$$

$$\text{subject to } v_1 + v_2 + \cdots + v_{20} = 1$$

Lecture 11 used the true Σ, and SAA uses $\hat{\Sigma}$.

Short selling

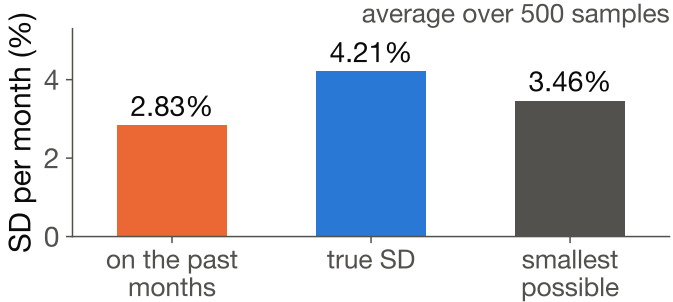


A weight below 0 is called short selling.

In one sample, 9 of the 20 weights are below 0.

With the true Σ, only 4 weights are below 0.

The SD on the past months is too low

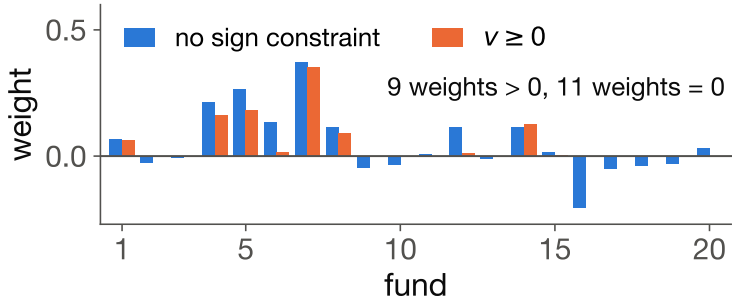


On the past months, the SD is 2.83% on average.

The true SD of the same portfolios is 4.21%.

The smallest possible SD is 3.46%.

No short selling



We add the constraint $v \geq 0$.

In the same sample, 11 of the 20 weights are now 0.

On average, the true SD falls from 4.21% to 3.82%.

Shrinking the covariance (Lecture 4)

$$\hat{\Sigma} + \lambda I$$



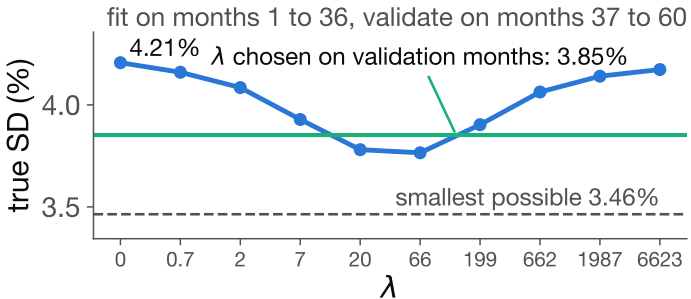
$\lambda = 0$:
the SAA portfolio

λ large:
equal weights

Adding λI to $\hat{\Sigma}$ is called shrinking the covariance.

Ridge in Lecture 4 also added λI.

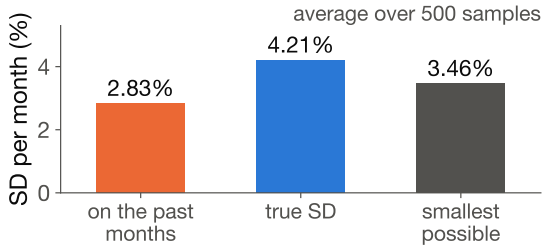
Choosing λ on validation months



As in Lecture 4, we choose λ on validation months.

On average, the true SD falls from 4.21% to 3.85%.

Summary: SAA for a portfolio



SAA uses the sample covariance matrix $\hat{\Sigma}$.

The SD on the past months is too low.

The constraint $v \geq 0$ or shrinking lowers the true SD.

Checkpoint 4

$$\hat{\Sigma} + \lambda I$$

We now have 6000 months for the same 20 funds.

Should λ be large or small?

Checkpoint 4

$$\hat{\Sigma} + \lambda I$$

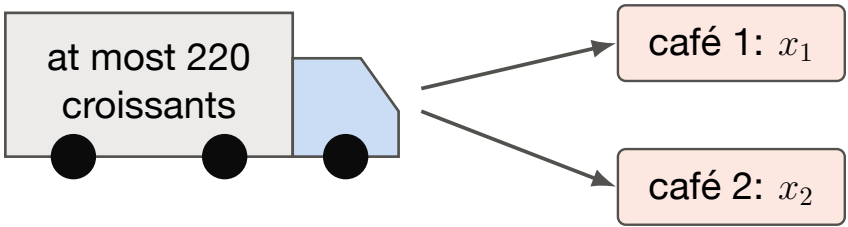
We now have 6000 months for the same 20 funds.

Should λ be large or small?

Small. With 6000 months, $\hat{\Sigma}$ is close to Σ.

So we need little shrinking.

Two cafés and one truck

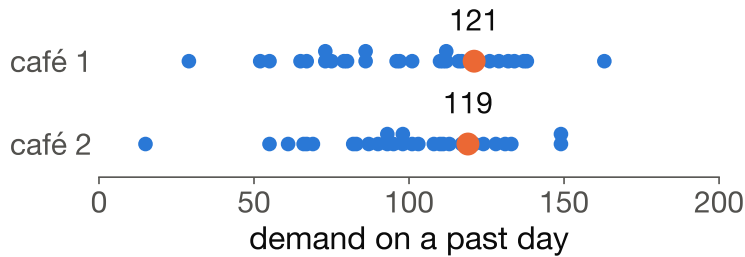


One truck brings croissants to two cafés.

The truck holds at most 220 croissants.

So the orders must satisfy $x_1 + x_2 \leq 220$.

Each café's SAA order

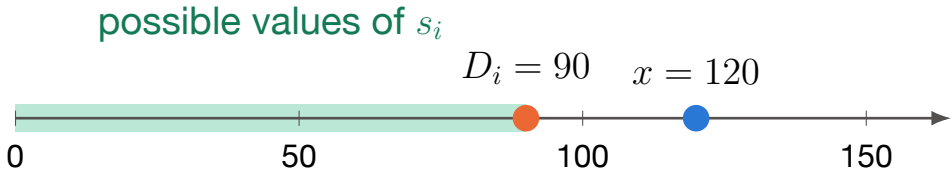


Each café has its own 30 past days.

Their SAA orders are 121 and 119.

The sum $121 + 119 = 240$ is more than 220.

Sales on a past day



Let s_i be the sales of a café on past day i.

$$s_i \leq x \text{ and } s_i \leq D_i$$

Each sale earns 80 baht, so the best s_i is $\min(D_i, x)$.

SAA with the truck is an LP (Lecture 8)

$$\text{maximize } \frac{1}{30}\sum_{i=1}^{30} 80(s_{i1}+s_{i2}) - 20(x_1+x_2)$$

$$\text{subject to } \underbrace{s_{ik}}_{\text{sales of café } k \text{ on day } i} \leq x_k,\ s_{ik} \leq D_{ik},\ x_1 + x_2 \leq 220$$

The objective and the constraints are linear.

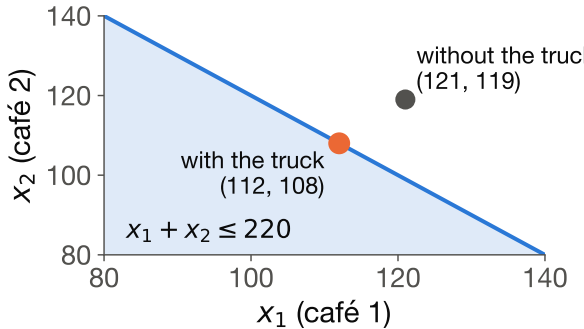
The truck LP in Python

```
import cvxpy as cp
x = cp.Variable(2)          # orders
s = cp.Variable((30, 2))    # sales
profit = 80 * cp.sum(s) / 30 - 20 * cp.sum(x)
cons = [s <= x, s <= D, cp.sum(x) <= 220]
cp.Problem(cp.Maximize(profit), cons).solve()
x.value    # [112, 108]
```

The table D holds 30 past demands for each café.

cvxpy gives orders of 112 and 108.

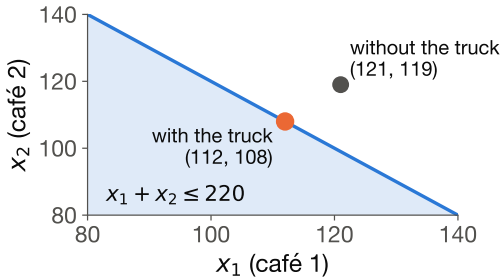
The orders with the truck



With the truck, the orders are 112 and 108.

The sum $112 + 108 = 220$ uses the whole truck.

Summary: SAA with a constraint



SAA with the truck constraint is an LP.

The sales on each past day are decision variables.

Checkpoint 5



The truck now holds at most 300 croissants.

What orders does the LP give?

Checkpoint 5



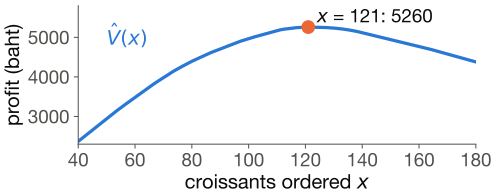
The truck now holds at most 300 croissants.

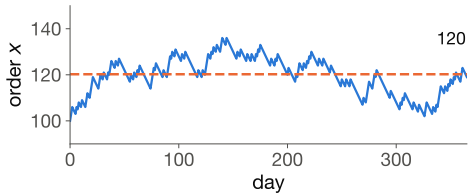
What orders does the LP give?

The sum $121 + 119 = 240$ is at most 300.

So the orders are 121 and 119.

Lecture summary



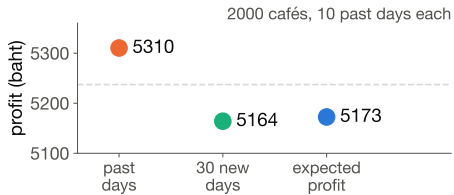


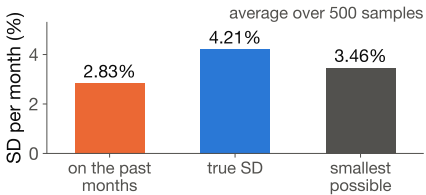
SAA maximizes the average profit on past days.

SGD changes the order one day at a time.

With a truck constraint, SAA is an LP.

Lecture summary (continued)



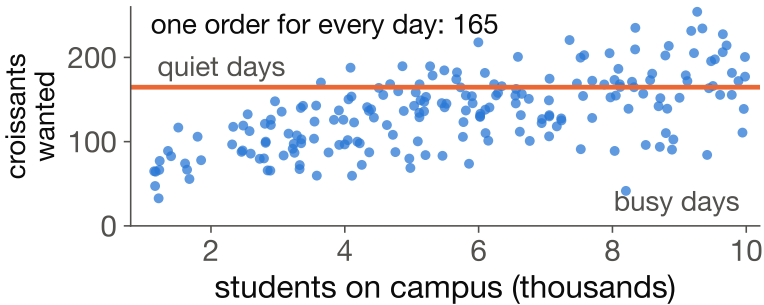


Profit on the past days is too high on average.

The SD on the past months is too low.

So we evaluate decisions on new data.

Next lecture



Today, the café used one order for every day.

In Lecture 13, busy days get a larger order.

Exercises

demand on 10 past days

70	85	90	95	100	110	120	125	140	160

1. Find the SAA order for $c = 30$ and $q = 90$.
2. With $\eta = 0.1$, $x = 100$ and $D = 130$, find the new x.
3. The truck holds 250 croissants. Find the orders.