Lecture 25: November 12

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25.1 Hypothesis Testing

In many statistical analyses, it is often the case that we want to make some assertion or conjecture about the distribution. This is where a *statistical hypothesis* has been made and we want to find a rule which helps us to decide whether to accept to reject the hypothesis. Typically, we want to perform a test on whether the unknown parameter of interest θ lies in set Ω_0 or Ω_1, which can be written as two competing hypotheses: a *null hypothesis* H_0 and an *alternative hypothesis* H_1:

$$H_0 : \theta \in \Omega_0 \quad \text{versus} \quad H_1 : \theta \in \Omega_1,$$

where $\Omega_0 \cap \Omega_1 = \emptyset$. We want to accept one of the hypotheses by inferring from a random sample X. Usually, a decision is made after we observe whether X lies in a set S or not.

$$H_0 \text{ is accepted and } H_1 \text{ is rejected when } X \notin S$$
$$H_1 \text{ is accepted and } H_0 \text{ is rejected when } X \in S.$$

Then we call S the *critical region* and S^c the *acceptance region*. We want to measure the performance of this test by the chance of falsely rejecting H_0 given that $\theta \in \Omega_0$. This is called the *power function*.

$$\beta(\theta) = \mathbb{P}_\theta(X \in S).$$

We want θ to be as close to 0 as possible for $\theta \in \Omega_0$ and as close to 1 as possible for $\theta \in \Omega_1$. In practice, we focus on minimizing the chance of error when H_0 is correct which can be measured by the *significance level*:

$$\alpha = \sup_{\theta \in \Omega_0} \beta(\theta) = \sup_{\theta \in \Omega_0} \mathbb{P}_\theta(X \in S).$$

25.2 Simple vs Simple hypothesis testing

A hypothesis is called *simple* if it specifies a single parameter. We consider a case where both hypotheses are simple.

$$H_0 : \theta = \theta_0$$
$$H_1 : \theta = \theta_1.$$

In this case, the power function has two values

$$\beta_0(x) = \mathbb{P}_0(\text{reject } H_0)$$
$$\beta_1(x) = \mathbb{P}_1(\text{reject } H_0).$$

Thus we want to minimize $\beta_0(x)$ and maximize $\beta_1(x)$. The statistic that we will use to quantify this test is the *likelihood ratio statistic*.

$$L(x) = \frac{\mathcal{L}(\theta_1|x)}{\mathcal{L}(\theta_0|x)}.$$

The acceptance region can then be traced by thresholding this statistic. The likelihood ratio test $L_c(x)$ (also called the Neyman-Pearson test) at the threshold c is given by

$$\begin{aligned}&\text{Accept } H_0 \text{ if } L(x) < c\\&\text{Reject } H_0 \text{ with probability } \gamma \text{ if } L(x) = c\\&\text{Reject } H_0 \text{ if } L(x) > c.\end{aligned}$$

The following celebrated Lemma from Neyman and Pearson shows that $L(x)$ is the *most powerful* test.

Lemma 25.1 (Neyman-Pearson Lemma)**.** Given any level $\alpha \in [0,1]$, there exists $c \geq 0$ such that the likelihood ratio test $L_c(x)$ has level α i.e. its decision gives $\beta_0(x) = \alpha$. Moreover, $L_c(x)$ is the *most powerful* test in a sense that it also maximizes $\beta_1(x)$ among all tests with level at most α.

The proof is omitted here, but we will be giving a couple of examples on how to utilize this lemma.

Example 25.2. Suppose that we sample X from a normal population and we have already deduced that it came from either $N(0,1)$ or $N(1,1)$. We will perform a Likelihood ratio hypothesis test at level α for the parameter $\mu = \mathbb{E}[X]$:

$$\begin{aligned}H_0 &: \mu = 0\\H_1 &: \mu = 1.\end{aligned}$$

First, we compute the likelihood ratio:

$$L(x) = \frac{\mathcal{L}(1|x)}{\mathcal{L}(0|x)} = \frac{(2\pi)^{-1/2}\exp\left(-(x-1)^2/2\right)}{(2\pi)^{-1/2}\exp\left(-x^2/2\right)} = \exp\left(-\frac{(x-1)^2}{2} + \frac{x^2}{2}\right) = \exp\left(\frac{2x-1}{2}\right).$$

Now we want to find the threshold $c \geq 1$ such that $\beta_0(x) = \alpha$. Note that

$$L(x) = \exp\left(\frac{2x-1}{2}\right) > c \text{ if and only if } x > \frac{2\log c + 1}{2}.$$

The critical region is then defined by this range of x, which leads us to solving for c in the following equation:

$$\begin{aligned}\mathbb{P}_0\left(X > \frac{2\log c + 1}{2}\right) &= \alpha\\1 - \mathbb{P}_0\left(X \leq \frac{2\log c + 1}{2}\right) &= \alpha\\\int_{-\infty}^{(2\log c+1)/2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx &= 1-\alpha,\end{aligned}$$

which can be solved numerically. For example, with $\alpha = 0.05$ we get $(2\log c + 1)/2$ resulting in $c = 0.117$. In practice, we do not need to solve for c since the critical region is defined by $x > c' = (2\log c + 1)/2 = -1.645$ where c' solves

$$\int_{-\infty}^{c'} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1 - \alpha = 0.95,$$

which can be looked up from the table directly.Thus a most powerful test is given by

$$\begin{aligned}&\text{Accept } H_0 \text{ if } x < c'\\&\text{Reject } H_0 \text{ if } x > c',\end{aligned}$$

and we can choose to either accept or reject H_0 when $x = c'$ since this will not change the value of β_0.

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Example 25.3. Suppose that X is a random sample with density

$$f(x|\theta) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

We want to test a hypothesis $H_0 : \theta = 1$ against $H_1 : \theta = \theta_1$ where $\theta_1 > 1$ is known. Using the likelihood ratio test, H_0 is rejected if

$$L(x) = \frac{\mathcal{L}(\theta_1|x)}{\mathcal{L}(1|x)} = \frac{\theta_1 e^{-\theta_1 x}}{e^{-x}} > c,$$

which is equivalent to

$$x < c' = \frac{\log(\theta_1/c)}{\theta_1 - 1}.$$

To make a test with level α, we solve

$$\begin{aligned} \beta_0(x) = \mathbb{P}_1(X < c') &= \alpha \\ \int_0^{c'} e^{-x}\, dx &= \alpha \\ 1 - e^{-c'} &= \alpha \\ c' &= -\log(1-\alpha). \end{aligned}$$

Thus a most powerful test is given by

$$\begin{aligned} &\text{Accept } H_0 \text{ if } x > -\log(1-\alpha) \\ &\text{Reject } H_0 \text{ if } x < -\log(1-\alpha), \end{aligned}$$

and we can choose to either accept or reject H_0 when $x = -\log(1-\alpha)$ since this will change the value of β_0. ◇

Example 25.4. Suppose X is a random sample from $\text{Bin}(2, p)$. We would like to make a test of $H_0 : p = 1/2$ against $H_1 : p = 3/4$ with significance level $\alpha = 0.05$. First, we compute the likelihood ratio

$$L(x) = \frac{\mathcal{L}(3/4|x)}{\mathcal{L}(1/2|x)} = \frac{\binom{2}{x}(3/4)^x(1/4)^{2-x}}{\binom{2}{x}(1/2)^x(1/2)^{2-x}} = \frac{3^x}{4}.$$

Thus we have

$$\begin{aligned} L(0) = 1/4 &\quad \text{with} \quad \mathbb{P}_0(0) = 1/4 \\ L(1) = 3/4 &\quad \text{with} \quad \mathbb{P}_0(0) = 1/2 \\ L(2) = 9/4 &\quad \text{with} \quad \mathbb{P}_0(0) = 1/4. \end{aligned}$$

We want to find a threshold c such that $L(x) > c$ implies $\beta_0(x) \leq 0.05$. Notice that if $X = 2$ is in the critical set i.e. $c < 9/4$, then $\beta_0(x) \geq \mathbb{P}_0(2) = 1/4$. On the other hand, if the critical set is empty i.e. $c > 9/4$, then $\beta_0(x) = 0$. Hence, it must be the case that $k = 9/4$. Now we need to solve for γ in

$$\begin{aligned} \beta_0(x) = \mathbb{P}_0(2)\mathbb{P}(\text{reject } H_0) = \frac{1}{4}\gamma &= 0.05 \\ \gamma &= \frac{1}{5}. \end{aligned}$$

Thus a most powerful test at level α is given by

$$\begin{aligned} &\text{Accept } H_0 \text{ if } L(x) < 9/4 \\ &\text{Reject } H_0 \text{ with probability } 1/5 \text{ if } L(x) = 9/4 \\ &\text{Reject } H_0 \text{ if } L(x) > 9/4. \end{aligned}$$

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