208711 Statistical Theory 1 **Fall 2019**

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26.1 One-sided hypothesis test

Now we consider a more general case.

$$H_0 : \theta \le \theta_0$$
$$H_1 : \theta > \theta_0.$$

The power function of this test is defined by

$$\beta_\theta = \mathbb{P}_\theta(\text{test rejects } H_0)$$

and the level of this test is defined by

$$\alpha = \sup_{\theta \le \theta_0} \beta_\theta.$$

The test is called *uniformly most powerful (UMP)* if

$$\beta_\theta \ge \tilde{\beta}_\theta \quad \text{for all } \theta \ge \theta_0$$

for any other test whose power function is $\tilde{\beta}$. We are looking at a specific kind of densities that allow us to easily obtain a UMP test on a statistic T. This is where we utilize the likelihood ratio statistic.

Definition 26.1. A family of densities $\{f(x|\theta)|\theta \in \Omega\}$ has *monotone likelihood ratios* if there exists a statistic $T = T(X)$ such that for any $\theta_1 < \theta_2$, the likelihood ratio $\mathcal{L}(\theta_2|x)/\mathcal{L}(\theta_1|x)$ is a nondecreasing function of $T(x)$. Here, we make a convention that $a/0 = \infty$ for any $a > 0$.

The situation $0/0$ can only arise on a null set and so it will be excluded from our discussion.

Example 26.2. Consider the family of normal distributions $N(\mu, \sigma^2)$. For any $\mu_1 < \mu_2$, the likelihood ratio from n random samples is

$$\begin{aligned}\frac{L(\mu_2|\boldsymbol{x})}{L(\mu_1|\boldsymbol{x})} &= \frac{(2\pi\sigma^2)^{n/2}e^{-\sum_i(x_i-\bar{x})^2/(2\sigma^2)}e^{-n(\bar{x}-\mu_2)^2/(2\sigma^2)}}{(2\pi\sigma^2)^{n/2}e^{-\sum_i(x_i-\bar{x})^2/(2\sigma^2)}e^{-n(\bar{x}-\mu_1)^2/(2\sigma^2)}}\\ &= e^{n\bar{x}(\mu_2-\mu_1)/\sigma^2}e^{n(\mu_1^2-\mu_2^2)/(2\sigma^2)}\end{aligned}$$

which is an increasing function of $T(x) = \bar{x}$. ⋄

In fact, it can be shown that any exponential family $f(x|\theta) = h(x)g(\theta)e^{\eta(\theta)T(x)}$ with increasing η has monotone likelihood ratios, which means that the Bernoulli, binomial, Poisson, beta and gamma distributions all have this property.

Suppose that we have such family of densities, a UMP test can then be defined with T statistic.

Theorem 26.3. Suppose that the family $\{f(x|\theta)|\theta \in \Omega\}$ has monotone likelihood ratios. Then the test S defined by

$$\begin{aligned} &\text{Accept } H_0 \quad \text{if } T(x) < c \\ &\text{Reject } H_0 \text{ with probability } \gamma \quad \text{if } T(x) = c \\ &\text{Reject } H_0 \quad \text{if } T(x) > c \end{aligned}$$

is a UMP test for the hypotheses $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$ with level $\alpha = \mathbb{P}_{\theta_0}(\text{reject } H_0)$. The values of c and γ can be adjusted to obtain a test at significance level α.

An analogous result also holds for the test $H_0 : \theta \geq \theta_0$ against $H_1 : \theta < \theta_0$ by changing sides of all inequalities in the theorem.

Proof: First, we will proof that for any $\theta_1 < \theta_2$ we have $\beta_{\theta_1} < \beta_{\theta_2}$. Consider the likelihood ratio

$$L(x) = L(T(x)) = \frac{\mathcal{L}(\theta_2|x)}{\mathcal{L}(\theta_1|x)}$$

which is a nondecreasing function of $T(x)$. Let $k = L(c)$. Then the test can be factored through L to obtain a likelihood ratio test:

$$\begin{aligned} &\text{Accept } H_0 \quad \text{if } L(x) < k \\ &\text{Reject } H_0 \quad \text{if } L(x) > k \end{aligned}$$

which is a UMP for the hypotheses $\theta = \theta_1$ against $\theta = \theta_2$. By letting $\tilde{S}$ be another test defined by randomly rejecting H_0 with probability $\alpha_1 = \beta_{\theta_1}$ and $\tilde{\beta}$ be its power function, we have

$$\beta_{\theta_2} \geq \tilde{\beta}_{\theta_2} = \mathbb{P}_{\theta_2}(\text{reject } H_0 \text{ with } \tilde{S} \text{ test}) = \alpha_1 = \beta_{\theta_1}.$$

as desired. Therefore, the level of the test S is

$$\sup_{\theta \leq \theta_0} \beta_\theta = \beta_{\theta_0} = \alpha.$$

Let S^* be another test for H_0 versus H_1 with power function β^*_θ and level at most α i.e. $\sup_{\theta \leq \theta_0} \beta^*_\theta \leq \alpha$. For any $\theta_1 > \theta_0$, we can restrict S and S^* to a simple hypothesis test $\theta = \theta_0$ against $\theta = \theta_1$, turning the former into a likelihood ratio test. Hence, by the Neyman-Pearson Lemma,

$$\beta_{\theta_1} \geq \beta^*_{\theta_1} \qquad \text{for all } \theta_1 > \theta_0,$$

which implies that S is a UMP test. ■

Example 26.4. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. From Example 26.2, a test that rejects $H_0 : \mu \leq \mu_0$ in favor of $H_1 : \mu > \mu_0$ if and only if $\bar{x} > c$ is a UMP. For example, if $\mu_0 = 0$ and $\alpha = 0.05$, the value of c can be found by solving

$$\begin{aligned} 0.05 = \alpha &= \mathbb{P}_0(\text{reject } H_0) \\ &= \mathbb{P}_0(\overline{X} > c) \\ &= 1 - \mathbb{P}_0(\overline{X} \leq c) \\ &= 1 - F(c), \end{aligned}$$

where F is the cdf of $N(0, \sigma^2/n)$. From this, we get $c = \sigma z_{0.05}/\sqrt{n}$, where $z_{0.05}$ is defined such that $\mathbb{P}(Z > z_{0.05}) = 0.05$ for $Z \sim N(0,1)$. ⋄

Example 26.5. Consider a two-sided hypothesis test $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$. If a UMP test at level α exists, then at any $\mu_1 > 0$, the power function β_{μ_1} for the following simple test

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu = \mu_1. \end{aligned}$$

is uniformly maximal. Let Γ be the critical region for S. As in Example 26.4, a UMP test obtained from Neyman-Pearson Lemma, denoted as S_1, has the critical region

$$\Gamma_1 = \left\{ X | \overline{X} > \frac{\sigma z_\alpha}{\sqrt{n}} \right\}.$$

If $\Gamma \neq \Gamma_1$ (a.e.) then we can construct a new test with the critical region $\Gamma' = \Gamma \cup \Gamma_1$. This test has level at most α and is a strictly better test than S_1, which is not possible since S_1 is UMP. Hence, $\Gamma = \Gamma_1$ (a.e.). On the other hand, for $\mu_2 < 0$, a UMP test for

$$\begin{aligned} H_0 &: \mu = 0 \\ H_1 &: \mu = \mu_2. \end{aligned}$$

obtained from the Neyman-Pearson Lemma has the critical region

$$\Gamma_2 = \left\{ X | \overline{X} < -\frac{\sigma z_\alpha}{\sqrt{n}} \right\}.$$

Applying the same argument as before, we must have that $\Gamma_1 = \Gamma_2 = \Gamma$ (a.e.), which again contradicts the uniqueness of the critical region. We conclude that the two-sided hypothesis test has no UMP test. $\diamond$