208711 Statistical Theory 1 **Fall 2019**

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28.1 Confidence sets and intervals

Definition 28.1. A set $C_n = C_n(X_1, X_2, \ldots, X_n)$ constructed from a random sample $X_1, X_2, \ldots, X_n$ is a $1-\alpha$ confidence interval if

$$\mathbb{P}_\theta(\theta \in C_n) \geq 1 - \alpha \qquad \text{for all } \theta.$$

If C_n is an interval i.e.

$$C_n = [L_n(X_1, X_2, \ldots, X_n), U(X_1, X_2, \ldots, X_n)]$$

we call C_n a *confidence* interval.

Example 28.2. Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma)$ with σ^2 known. Let $L_n = \overline{X} - c$ and $U_n = \overline{X} + c$. Then

$$\begin{aligned}
\mathbb{P}_\mu(\mu \in [L_n, U_n]) &= \mathbb{P}_\mu\left(L_n \leq \mu \leq U_n\right) \\
&= \mathbb{P}_\mu\left(-c \leq \overline{X} - \theta \leq c\right) \\
&= \mathbb{P}_\mu\left(-\frac{c}{\sigma/\sqrt{n}} \leq \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leq \frac{c}{\sigma/\sqrt{n}}\right) \\
&= \mathbb{P}_\mu\left(-\frac{c\sqrt{n}}{\sigma} \leq Z \leq \frac{c\sqrt{n}}{\sigma}\right) \\
&= 1 - 2F(-c\sqrt{n}/\sigma)
\end{aligned}$$

where F is the cdf of $N(\mu, \sigma)$. If we choose $c = cz_{\alpha/2}/\sqrt{n}$ then $\mathbb{P}_\mu(\mu \in C_n) = 1 - \alpha$. ◇

Example 28.3. Let $X_i \sim N(\mu_i, \sigma^2)$ with σ^2 known. Then

$$\|\boldsymbol{X} - \boldsymbol{\theta}\|^2 = \sum_i (X_i - \theta_i)^2 \sim \chi^2_n.$$

Let $C_n = \{\boldsymbol{\theta} : \|\boldsymbol{X} - \boldsymbol{\theta}\| > c\}$. Then

$$\mathbb{P}_{\boldsymbol{\theta}}(\boldsymbol{\theta} \in C_n) = 1 - \mathbb{P}_{\boldsymbol{\theta}}(\|\boldsymbol{X} - \boldsymbol{\theta}\| \leq c) = 1 - \mathbb{P}_{\boldsymbol{\theta}}(\chi^2_n \leq c).$$

If $c = F^{-1}(\alpha)$ where F is the cdf of χ^2_n, then S_n is a $1-\alpha$ confidence set. ◇

28.2 Inverting a test

Consider a level α test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Define the acceptance region by

$$A(\theta_0) = \{(X_1, X_2, \ldots, X_n) : \text{ accept } H_0\}$$

and the confidence set

$$C = C(x_1, x_2, \ldots, x_n) = \{\theta : (x_1, x_2, \ldots, x_n) \in A(\theta)\}.$$

Since the probability of accepting H_0 when $\theta = \theta_0$ is α,

$$\mathbb{P}_{\theta_0}(\theta_0 \in C) = \mathbb{P}_{\theta_0}((X_1, X_2, \ldots, X_n) \in A(\theta_0)) = \mathbb{P}_{\theta_0}(\text{accept } H_0) = 1 - \alpha.$$

Thus C is a $1-\alpha$ confidence interval. On the other hand, if C is a confidence set, then we can construct a level α test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ as follows:

$$\begin{aligned} &\text{Accept } H_0 \quad \text{if } \theta_0 \in C(x_1, x_2, \ldots, x_n) \\ &\text{Reject } H_0 \quad \text{if } \theta_0 \notin C(x_1, x_2, \ldots, x_n). \end{aligned}$$

Example 28.4. Consider the test $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$. Let $\hat{\theta}$ be the MLE of θ then the likelihood ratio test rejects H_0 when

$$\frac{\mathcal{L}(\theta_0|x)}{\mathcal{L}(\hat{\theta}|x)} \leq c.$$

Thus, the confidence region is $C = \{\mathcal{L}(\theta)/\mathcal{L}(\hat{\theta})\} > c$. ⋄

Example 28.5. Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma)$ with σ^2 known. The likelihood ratio test of $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ rejects H_0 at level α when

$$|\overline{X} - \mu_0| < \frac{\sigma}{\sqrt{n}} z_{\alpha/2}.$$

Thus the confidence set is

$$C = \left\{\mu : |\overline{X} - \mu| < \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right\} = \left[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right].$$

If σ^2 is unknown, we obtain the confidence set from the Wald's test:

$$C = \left[\overline{X} - \frac{S}{\sqrt{n}} t_{n-1,\alpha/2}, \overline{X} + \frac{S}{\sqrt{n}} t_{n-1,\alpha/2}\right].$$

⋄

28.3 Pivots

In each of the previous examples, we have inadvertantly used something that led us to obtain the confidence interval that does not depend on the parameter. This is called a *pivot*.

Definition 28.6. A function $Q(X_1, X_2, \ldots, X_n, \theta$ of a random sample and the parameter is a *pivot* if its distribution does not depend on θ.

Example 28.7. Let $X_1, X_2, \ldots, X_n \sim N(\mu, 1)$. Then $Q = \overline{X} - \mu \sim N(0, 1/n)$ is independent of μ. ⋄

If such Q exists, for any $0 \leq a \leq 1$, we can find $a \geq -\infty$ and $b \leq \infty$ such that

$$\mathbb{P}_\theta(a \leq Q(\boldsymbol{X}, \theta) \leq b) \geq 1 - \alpha$$

for all θ, which allows us to obtain a $1-\alpha$ confidence inter val $C = \{\theta : a \leq Q(\boldsymbol{X}, \theta) \leq b\}$.

Example 28.8. Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma)$ with σ^2 known. Then

$$Z = \frac{\sqrt{n}(\overline{X} - \mu)}{\sigma} \sim N(0, 1)$$

is a pivot and the choice of $a = -z_{\alpha/2}$ and $b = z_{\alpha/2}$ allows us to obtain the $1-\alpha$ confidence interval, that is

$$\begin{aligned} C &= \{\mu : -z_{\alpha/2} \le Z \le z_{\alpha/2}\} \\ &= \left\{\mu : \frac{\sigma}{\sqrt{n}} z_{\alpha/2} \le |\overline{X} - \mu| \le \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right\} \\ &= \left[\overline{X} - \frac{\sigma}{\sqrt{n}} z_{\alpha/2}, \overline{X} + \frac{\sigma}{\sqrt{n}} z_{\alpha/2}\right]. \end{aligned}$$

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Example 28.9. Let $X_1, X_2, \ldots, X_n \sim unif(0, \theta)$. Let $Q = \max_i X_i/\theta$. Then $0 \le Q \le 1$ and we have that

$$\mathbb{P}_\theta(Q \le c) = \prod_{i=1}^{n} \mathbb{P}_\theta(X_i \le c\theta) = c^n.$$

To obtain a $1-\alpha$ confidence interval, we let $c = \alpha^{1/n}$. Then $\mathbb{P}_\theta(Q \le c) = \alpha$ and it follows that

$$\begin{aligned} 1 - \alpha &= \mathbb{P}_\theta(c \le Q \le 1) \\ &= \mathbb{P}_\theta\left(c \le \frac{\max_i X_i}{\theta} \le 1\right) \\ &= \mathbb{P}_\theta\left(c \ge \frac{\theta}{\max_i X_i} \ge 1\right) \\ &= \mathbb{P}_\theta\left(\max_i X_i \le \theta \le \frac{\max_i X_i}{c}\right). \end{aligned}$$

Thus a $1-\alpha$ confidence interval is

$$\left(\max_i X_i, \frac{\max_i X_i}{\alpha^{1/n}}\right)$$

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