208711 Statistical Theory 1	**Fall 2021**

Homework 5: October 22

Turn in your solutions for problem 1,2,3,5,6 and 8.

1. Let $X, X_1, X_2, \ldots, X_n$ be iid from an exponential distribution with parameter λ.
   - Find the density of $Y = \lambda X$.
   - Let $\overline{X} = (X_1 + X_2 + \ldots + X_n)/n$. Show that $\overline{X}$ and $\left(X_1^2 + X_2^2 + \ldots, X_n^2\right)/\overline{X}^2$ are independent.
2. Let $X_1, X_2, \ldots, X_n \sim \text{Pois}(\lambda)$.
   - Find a complete sufficient statistic for λ. (Hint: write the pdf as an exponential family.
   - Find a UMVU estimator for λ.
3. Let $X_1, X_2, \ldots, X_n$ be iid with pdf

   $$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1, \quad 0 < \theta < \infty.$$

   Find the MLE of θ.
4. Let $X_1, \ldots, X_n$ be iid sample from $N(\mu, 1)$ and M_n be the median of $X_1, \ldots, X_n$. Assume that we know that $\mathbb{E}(M_n) = \mu$. Use the Rao-Blackwell theorem to show that the varince of $\overline{X}$ is less thant the variance of M_n. (Hint: write $M_n = M_n - \overline{X} + \overline{X}$ and show that $M_n - \overline{X}$ and $\overline{X}$ are independent.
5. Let $X_1, X_2, \ldots, X_n$ be an iid sample from unif$[0, \theta]$. Find the MLE of θ. (Hint: DO NOT use calculus to solve this problem; look at the graph of the joint pdf for different values of θ.)
6. One observation X is taken on a discrete random variable with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$ given in the table below. Find the MLE of θ.

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	$\frac{1}{3}$	$\frac{1}{4}$	0
1	$\frac{1}{3}$	$\frac{1}{4}$	0
2	0	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
4	$\frac{1}{6}$	0	$\frac{1}{4}$

7. Suppose that the random variables $Y_1, \ldots Y_n$ satisfy

   $$Y_i = \beta x_i + \epsilon_i, i = 1, \ldots, n,$$

   where $x_1, \ldots, x_n$ are fixed constants, and $\epsilon_1, \ldots, \epsilon_n$ are iid sample from $N(0\sigma^2)$ where σ^2 is unknown.

(a) Find a two-dimensional sufficient statistic for (β, σ^2). (Hint: write the joint pdf as an exponential family).

(b) Find the MLE of β and show that it is an unbiased estimator of β.

8. Let $X_1, \ldots, X_n$ be iid Poisson(λ) and let λ have a gamma(α, β) distribution, the conjugate family for Poisson.

    (a) Find the posterior distribution of λ.

    (b) Calculate the posterior mean and variance.