208711 Statistical Theory 1 **Fall 2021**

Homework 6: October 29

Turn in your solutions for problem 1,2,3.

1. Let X be a random variable whose probability mass function under H_0 and H_1 is given by

x	1	2	3	4	5	6	7
$f(x\|H_0)$	.01	.01	.01	.01	.01	.01	.94
$f(x\|H_1)$	.06	.05	.04	.03	.02	.01	.79

   Use the Neyman-Pearson Lemma to find a UMP test for H_0 against H_1 with level $\alpha = 0.04$ and compute the type II error $\mathbb{P}(\text{accept } H_0 | H_1 \text{ is true})$

2. Suppose X is one observation from a population with Beta$(\theta, 1)$ pdf.
   (a) For testing $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$, compute the power function $\beta(\theta)$ and the level $\max_{\theta \in H_0} \beta(\theta)$ of the test that rejects H_0 if $X > \frac{1}{2}$.
   (b) Find a UMP test of $H_0 : \theta \leq 1$ versus $H_1 : \theta > 1$.

3. Derive a $1 - \alpha$ confidence interval for a binomial $Y \sim$ binomial(n, p) by inverting likelihood ratio test of $H_0 : p = p_0$ versus $H_1 : p \neq p_0$. (The interval might be a complicated function of p, the MLE $\hat{p}$, n and Y.

4. Show that for a random sample $X_1, X_2, \ldots, X_n$ from a $N(0, \sigma^2)$ population, the most powerful test of $H_0 : \sigma = \sigma_0$ versus $H_1 : \sigma = \sigma_1$ where $\sigma_0 < \sigma_1$ is given by

$$\text{Accept } H_0 \quad \text{if } \sum_i X_i^2 \leq c$$
$$\text{Reject } H_0 \quad \text{if } \sum_i X_i^2 > c,$$

   and show that the test has α significance level if we set $c = \sigma_0^2 \chi^2_{n,\alpha}$.

5. Let $X_1, X_2, \ldots, X_n$ be iid Poisson(λ). Find a UMP test of $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$ at significance level α. The test should be written in terms of the cdf of Poisson$(n\lambda_0)$.

6. Let $X_1, X_2, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ where σ^2 is known. For each of the following hypotheses, write out the acceptance region of a level α test and the $1 - \alpha$ confidence interval that results from inverting the test.
   (a) $H_0 : \mu = \mu_0$ versus $H_0 : \mu \neq \mu_0$.
   (b) $H_0 : \mu \geq \mu_0$ versus $H_0 : \mu < \mu_0$.

7. Find a $1 - \alpha$ confidence interval for θ, given $X_1, \ldots, X_n$ with a pdf
   (a) $f(x|\theta) = 1, \quad \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$
   (b) $f(x|\theta) = 2x/\theta^2, \quad 0 < x < \theta, \quad \theta > 0.$