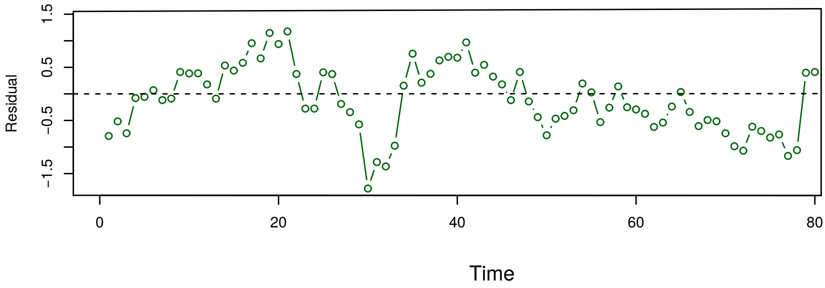
Time Series Analysis 1
DS351

Why can't we use linear regression

Simple model

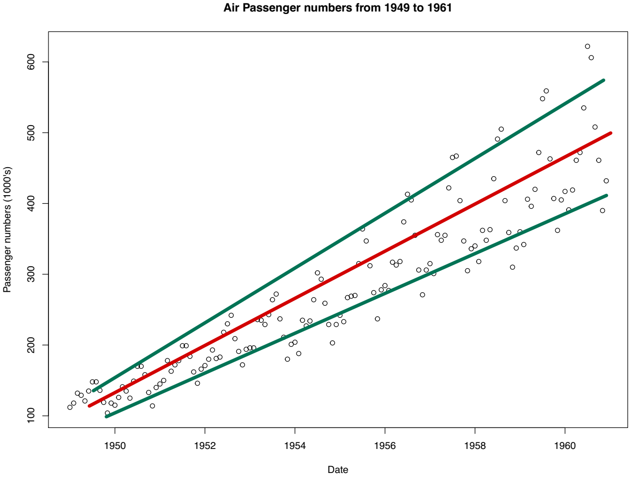
$$y_t = \beta_0 + \beta_1 t + \varepsilon_t$$



Error terms are correlated.

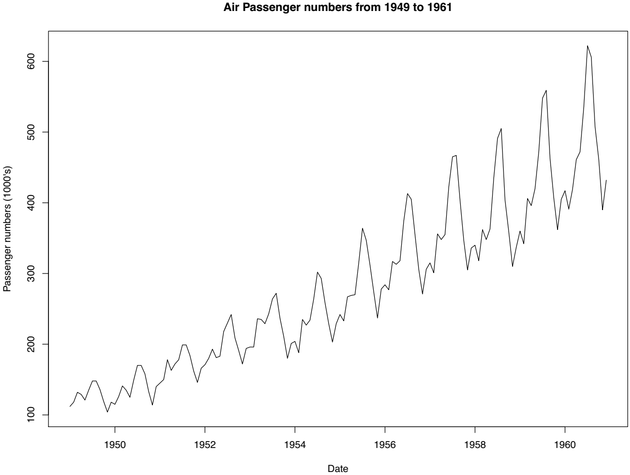
Why can't we use linear regression

quantile regression



Variance of the errors increases with time.

Why can't we use linear regression



Seasonality, which implies non-linearity!

Analyzing Time Series

Notations

Time series is often denoted by

$$\cdots \quad Y_{t-2} \quad Y_{t-1} \quad Y_t \quad Y_{t+1} \quad Y_{t+2} \quad \cdots$$

time index

Notations

Time series is often denoted by

$$\cdots \quad Y_{t-2} \quad Y_{t-1} \quad Y_t \quad Y_{t+1} \quad Y_{t+2} \quad \cdots$$

time index

Lag is an amount of time passed.

Example: lag 5 of Y_t is Y_{t-5}.

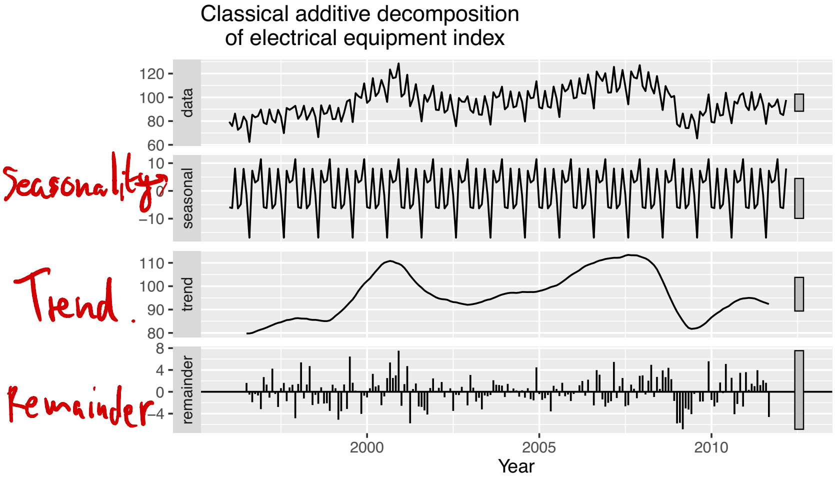
lag 5 of Y_{t+1} is Y_{t-4}

Time series decomposition

Time series decomposition

Goal:

- Extract trend seasonality
- Visualize and improve understanding of time series



Classical decomposition

Two types of decomposition:

1. Additive decomposition

$$y_t = S_t + T_t + R_t,$$

where

- S_t is the seasonal component.
- T_t is the trend component.
- R_t is the remainder component.

Classical decomposition

Two types of decomposition:

2. Multiplicative decomposition

$$y_t = S_t \times T_t \times R_t,$$

where

- S_t is the seasonal component.
- T_t is the trend component.
- R_t is the remainder component.

Additive decomposition

$$y_t = S_t + T_t + R_t,$$

Step 1: Estimate the **Trend** $\hat{T}_t$.

Moving average is a method to estimate the trend.

Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

Moving averages

$m = 5$

Moving average is a method to estimate the trend

$\boxed{y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5} \quad y_6 \quad y_7$

Time series: $y_t : y_1, y_2, \ldots, y_T$

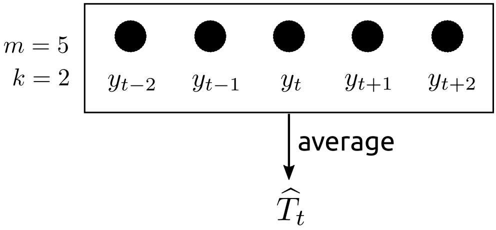
$\frac{y_1 + \ldots + y_5}{5} = \hat{T}_3, \hat{T}_4, \hat{T}_5$

Moving average of order *m* of y_t is

$$\widehat{T}_t = \frac{1}{m} \sum_{i=-k}^{k} y_{t+i},$$

Ex: $m = 5$

where $m = 2k + 1$.



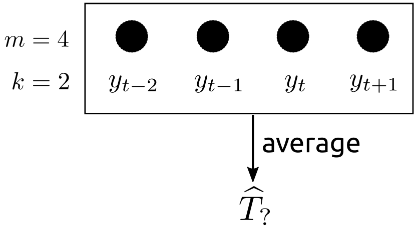
Example: electricity sold to customers in South Australia

Year	Sales (GWh)	5-MA	
1989	2354.34		
1990	2379.71		
1991	2318.52	2381.53	$\hat{T}_{1991}$
1992	2468.99	2424.56	$\hat{T}_{1992}$
1993	2386.09	2463.76	
1994	2569.47	2552.60	
1995	2575.72	2627.70	
1996	2762.72	2750.62	
1997	2844.50	2858.35	
⋮	⋮	⋮	⋮
~~1997~~ 2005	2844.50	2858.35	
2006	3527.48	3485.43	$\hat{T}_{2006}$
2007	3637.89		
2008	3655.00		

Example: moving average of different orders

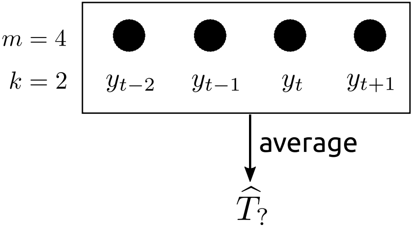
Moving average of even orders

For example, $m = 4$

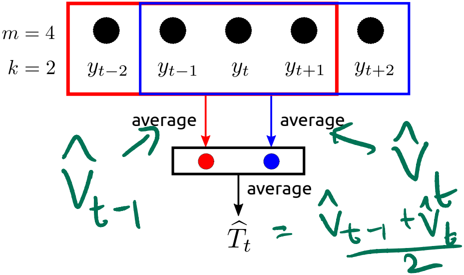


Moving average of even orders

For example, $m = 4$



Idea: use 2-MA **after** 4-MA



$$\widehat{T}_t = \frac{\hat{V}_{t-1} + \hat{V}_t}{2}$$

Australian quarterly beer production

$m = 4$

Year	Quarter	Observation	4-MA	2x4-MA
1992	Q1	443		
1992	Q2	410	$\boxed{451.25}$	
1992	Q3	420	$\boxed{448.75}$	$\boxed{450}$
1992	Q4	532	451.5	450.12
1993	Q1	433	449	450.25
1993	Q2	421	444	446.5
1993	Q3	410	448	446
1993	Q4	512	438	443
1994	Q1	449	441.25	439.62
⋮	⋮	⋮	⋮	⋮
1996	Q3	398	433.75	430.88
1996	Q4	507	433.75	433.75
1997	Q1	--	--	
1997	Q_2	--	--	

$\hat{T}_{1992,Q_3}$

2×m-MA

The 2×4-MA of y_t is

$$\begin{aligned}\widehat{T}_t &= \frac{1}{2}\left[\frac{1}{4}(y_{t-2} + y_{t-1} + y_t + y_{t+1}) + \frac{1}{4}(y_{t-1} + y_t + y_{t+1} + y_{t+2})\right] \\ &= \frac{1}{8}y_{t-2} + \frac{1}{4}y_{t-1} + \frac{1}{4}y_t + \frac{1}{4}y_{t+1} + \frac{1}{8}y_{t+2}.\end{aligned}$$

The 2×m-MA

$$\hat{T}_t = \frac{1}{2}\left[\frac{1}{m}\left(y_{t-\frac{m}{2}} + \ldots + y_{t+1}\right) + \frac{1}{m}\left(y_{t-\frac{m}{2}+1} + \ldots + y\right)\right]$$

$$= \frac{1}{2m}y_{t-\frac{m}{2}} + \frac{1}{m}y_{t-\frac{m}{2}+1} + \ldots + \frac{1}{m}y_{t+\frac{m}{2}-1} + \frac{1}{2m}y_{t+\frac{m}{2}}$$

2×m-MA

The 2×4-MA of y_t is

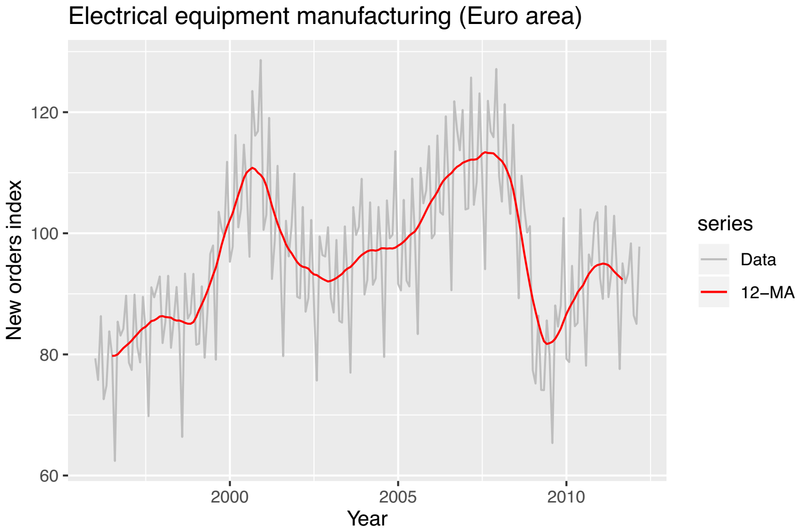
$$\begin{aligned}\widehat{T}_t &= \frac{1}{2}\left[\frac{1}{4}(y_{t-2}+y_{t-1}+y_t+y_{t+1})+\frac{1}{4}(y_{t-1}+y_t+y_{t+1}+y_{t+2})\right]\\ &= \frac{1}{8}y_{t-2}+\frac{1}{4}y_{t-1}+\frac{1}{4}y_t+\frac{1}{4}y_{t+1}+\frac{1}{8}y_{t+2}.\end{aligned}$$

In general, The 2×m-MA of y_t is

$$\widehat{T}_t = \frac{1}{2m}y_{t-k}+\ldots+\frac{1}{m}y_{t-1}+\frac{1}{m}y_t+\frac{1}{m}y_{t+1}+\ldots+\frac{1}{2m}y_{t+k},$$

where $m = 2k$.

Example: monthly data



2×4-MA for quarterly beer production, 7-MA for daily traffic data etc

Step 2: Calculate the detrended series

$$y_t - \widehat{T}_t$$

If $m=5$,

there is no

$\hat{T}_1$, $\hat{T}_2$,

$\hat{T}_n$, $\hat{T}_{n-1}$

Additive decomposition

Decompose "detrended series": $y_t - \hat{T}_t$

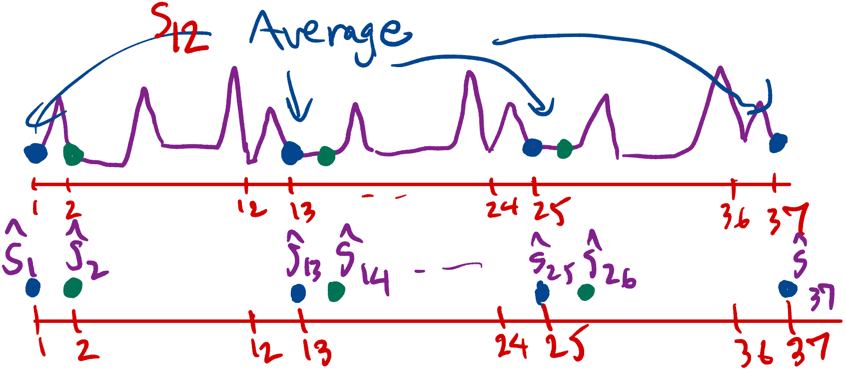
Step 3: Compute the mean of $y_t - \widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$S_1 = \text{the mean of all values in January}$$

$$S_2 = \text{the mean of all values in February}$$

$\vdots$ and so on...

S_{12}



Additive decomposition

Step 3: Compute the mean of $y_t - \widehat{T}_t$ for each seasonal unit.
For example, for monthly data, we compute

S_1 = the mean of all values in January $\Rightarrow \overline{S} = \frac{S_1 + \ldots + S_{12}}{12}$

S_2 = the mean of all values in February

$\vdots$ and so on...

S_{12}

Then, these seasonal values are adjusted to have zero mean.

Final value of seasonality $\rightarrow$ $\widehat{S}_1 = S_1 - \overline{S} = S_1 - \frac{S_1 + S_2 + \ldots + S_{12}}{12}$

$\widehat{S}_2 = S_2 - \overline{S} = S_2 - \frac{S_1 + S_2 + \ldots + S_{12}}{12}$

$\vdots$ and so on...

$\widehat{S}_{12} = S_{12} - \overline{S}$

where $\overline{S} = \frac{1}{12}\sum_{i=1}^{12} S_i$

$\widehat{S}_{25} = \widehat{S}_{13} = \widehat{S}_1, \widehat{S}_{14} = \widehat{S}_2, \ldots$

Additive decomposition

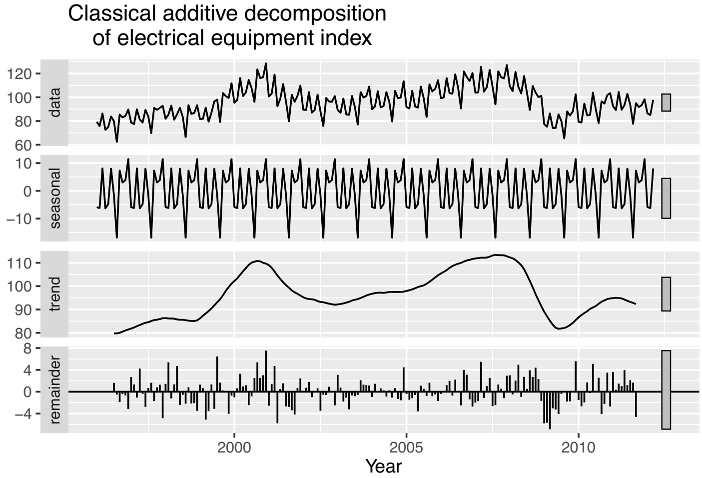
Step 4: The remainder component is

$$\widehat{R}_t = y_t - \widehat{T}_t - \widehat{S}_t.$$

$\hat{S}_t$

$\hat{T}_t$

$\hat{R}_t$



Multiplicative decomposition

$$Y_t = \hat{S}_t \times \hat{T}_t \times \hat{R}_t$$

Only used for positive data

- *Step 1*: Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

Multiplicative decomposition

- *Step 1*: Pick m, usually the seasonal period.

$$\widehat{T}_t = \begin{cases} m\text{-MA} & \text{if } m \text{ is an odd number.} \\ 2 \times m\text{-MA} & \text{if } m \text{ is an even number.} \end{cases}$$

- *Step 2*: Calculate the detrended series

$$\frac{y_t}{\widehat{T}_t}.$$

Multiplicative decomposition

- *Step 3*: Compute the mean of $y_t/\widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$
\begin{aligned}
S_1 &= \text{the mean of all values in January} \\
S_2 &= \text{the mean of all values in February} \\
&\text{and so on...}
\end{aligned}
$$

Multiplicative decomposition

- *Step 3*: Compute the mean of $y_t / \widehat{T}_t$ for each seasonal unit. For example, for monthly data, we compute

$$S_1 = \text{the mean of all values in January}$$
$$S_2 = \text{the mean of all values in February}$$
$$\vdots \quad \text{and so on...}$$
$$S_{12}$$

$$\hat{S}_1 + \hat{S}_2 + \dots + \hat{S}_{12} = 1.$$

Then, these seasonal values are adjusted to have sum of 1 .

why not

If $S_i - \bar{S} = 0$

$$\widehat{S}_1 = S_1 / \overline{S} = \frac{S_1}{S_1 + \dots + S_{12}}$$
$$\widehat{S}_2 = S_2 / \overline{S} = \frac{S_2}{S_1 + \dots + S_{12}}$$
$$\text{and so on...}$$
$$\hat{S}_{12} = S_{12} / \bar{S} = \frac{S_{12}}{S_1 + \dots + S_{12}}$$

$$\hat{S}_{13} = \hat{S}_1, \quad \hat{S}_{14} = \hat{S}_2, \dots$$

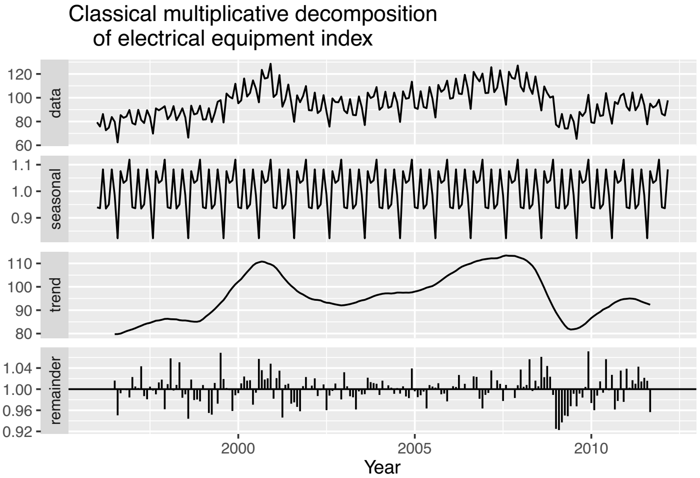
where $\overline{S} = \sum_{i=1}^{12} S_i$.

Multiplicative decomposition

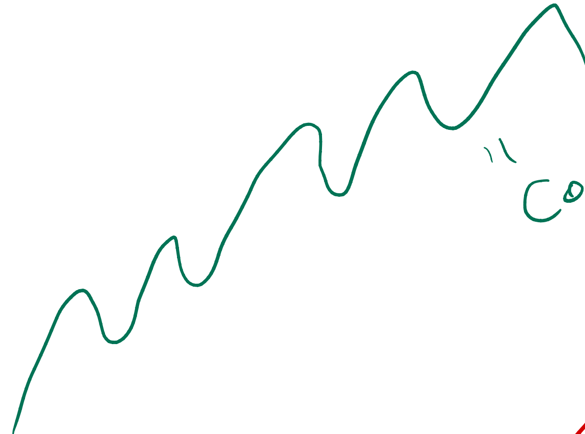
- *Step 4*: The remainder component is

$$\widehat{R}_t = \frac{y_t}{\widehat{T}_t \widehat{S}_t}.$$

always > 0

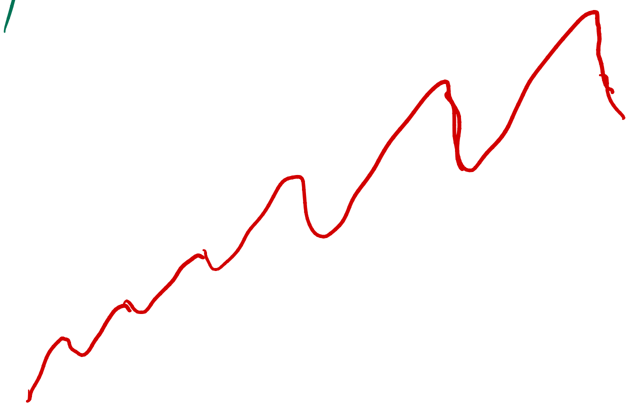


Additive:



"constant variance"

Multiplicative:



Strength of trend (Wang, Smith & Hyndman, 2006)

Back to additive decomposition:

$$y_t = T_t + S_t + R_t.$$

Observation: for a time series with strong trend,

$$\left(\frac{\mathrm{Var}(R_t)}{\mathrm{Var}(T_t + R_t)}\right) \text{ should be small.}$$

small if strong trend

Strength of trend (Wang, Smith & Hyndman, 2006)

Back to additive decomposition:

$$y_t = T_t + S_t + R_t.$$

Observation: for a time series with strong trend,

$$\frac{\text{Var}(R_t)}{\text{Var}(T_t + R_t)} \quad \text{should be small.}$$

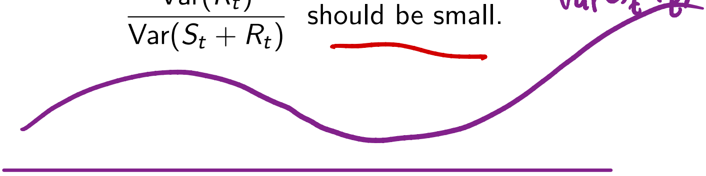
for a time series with strong seasonality,

$$\frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)} \quad \text{should be small.}$$

No seasonality

$$\Rightarrow \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)} = 1$$

Ex:



Strength of trend (Wang, Smith & Hyndman, 2006)

So we define the strength of trend as

$$F_T = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(T_t + R_t)}\right)$$

$= 0$ if $\text{Var}(T_t+R_t) < \text{Var}(R_t)$.

$F_T = 1 - \frac{\text{Var}(R_t)}{\text{Var}(T_t+R_t)}$ if $\text{Var}(T_t+R_t) > \text{Var}(R_t)$

and the strength of seasonality as

$$F_S = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)}\right).$$

Higher value = Stronger effect

This is useful when we have a collection of time series and we want to find the one with the most trend or seasonality.

Forecasting with decomposition

suppose last date is t

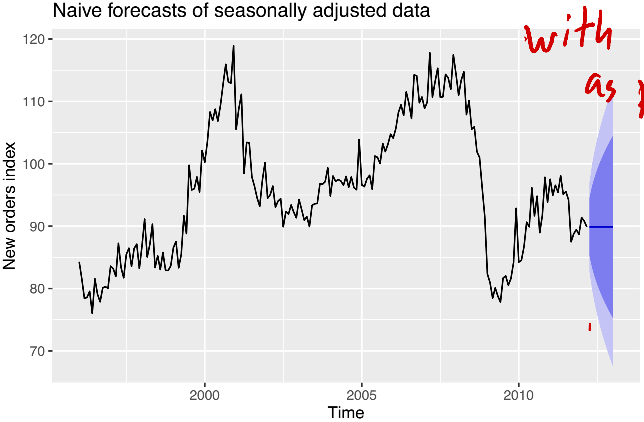
We can make forecast from the decomposition

$$y_t = \widehat{S}_t + (\widehat{T}_t + \widehat{R}_t),$$

$\hat{y}_{t+1} = \hat{S}_{t+1} + \hat{T}_{t+1}$

where we can use time series model to forecast the seasonally adjusted component $\widehat{A}_t = \widehat{T}_t + \widehat{R}_t$ and then add back the seasonal component S_t.

Use linear regression with day/month as predictor.



Forecasting with decomposition

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